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Ecole Doctorale Sciences, Technologie et Santé



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Large Solutions for Fractional Laplacian Operators

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Large Solutions for Fractional Laplacian Operators

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Résumé. *La thèse étudie les problèmes de Dirichlet linéaires et semilinéaires pour différents opérateurs du type Laplacien fractionnaire. Les données peuvent être des fonctions régulières ou plus généralement des mesures de Radon. Le but est de classifier les solutions qui présentent une singularité au bord du domaine prescrit.*

Nous remarquons d’abord l’existence de toute une gamme de fonctions harmoniques explosants au bord et nous les caractérisons selon une nouvelle notion de trace au bord. À l’aide d’une nouvelle formule d’intégration par parties, nous élaborons ensuite une théorie faible de type Stampacchia pour étendre la théorie linéaire à un cadre qui comprend ces fonctions : nous étudions les questions classiques d’existence, d’unicité, de dépendance à l’égard des données, la régularité et le comportement asymptotique au bord.

Puis, nous développons la théorie des problèmes semilinéaires, en généralisant la méthode des sous- et sursolutions. Cela nous permet de construire l’analogue fractionnaire des grandes solutions dans la théorie des EDPs elliptiques nonlinéaires, en donnant des conditions suffisantes pour l’existence.

La thèse se termine par la définition et l’étude d’une notion de courbures directionnelles nonlocales.

Mots clés: Laplacien fractionnaire, opérateurs nonlocaux, grandes solutions, solutions faibles L^1 , équations elliptiques nonlinéaires, problème de Dirichlet, singularité au bord, courbures nonlocales.

Abstract. *The thesis studies linear and semilinear Dirichlet problems driven by different fractional Laplacians. The boundary data can be smooth functions or also Radon measures. The goal is to classify the solutions which have a singularity on the boundary of the prescribed domain.*

We first remark the existence of a large class of harmonic functions with a boundary blow-up and we characterize them in terms of a new notion of degenerate boundary trace. Via some integration by parts formula, we then provide a weak theory of Stampacchia’s sort to extend the linear theory to a setting including these functions: we study the classical questions of existence, uniqueness, continuous dependence on the data, regularity and asymptotic behaviour at the boundary.

Afterwards we develop the theory of semilinear problems, by adapting and generalizing some sub- and supersolution methods. This allows us to build the fractional counterpart of large solutions in the elliptic PDE theory of nonlinear equations, giving sufficient conditions for the existence.

The thesis is concluded with the definition and the study of a notion of nonlocal directional curvatures.

Keywords: fractional Laplacian, nonlocal operators, large solutions, L^1 weak solutions, nonlinear elliptic equations, Dirichlet problem, boundary singularity, nonlocal curvatures.

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Ce manuscrit présente l'étude des solutions avec explosion au bord pour deux différents opérateurs intégro-différentiels, l'un et l'autre appelés *Laplacien fractionnaire*. Ce type d'opérateurs a été étudié intensivement dans les dernières années, parce qu'ils montrent beaucoup de ressemblances avec la théorie classique des opérateurs elliptiques, tandis qu'ils sont souvent définis comme opérateurs intégraux avec noyaux singuliers et interactions à longue portée, ce qui les rend des opérateurs *nonlocaux*¹. En même temps, ils apparaissent naturellement dans des applications concrètes, notamment pour modéliser des phénomènes où des interactions à longue portée sont présents. On a par exemple : modèles de transition de phase [79], dislocation des cristaux [84], problème de l'obstacle [78], propagation des flammes [80], surfaces minimales [17], finance [29], et sciences des matériaux [9].

Le Laplacien fractionnaire. On va appeler le premier de ces deux opérateurs simplement² *Laplacien fractionnaire* et on le dénote par $(-\Delta)^s$. Fixée $s \in (0, 1)$, pour une fonction³ $u : \mathbb{R}^N \rightarrow \mathbb{R}$ cet opérateur admet différentes définitions équivalentes et il peut être introduit :

- comme un opérateur intégral avec noyau singulier [37]

$$(-\Delta)^s u(x) = C_{N,s} \text{ p.v. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy$$

où “p.v.” dénote l'intégral au sens de la valeur principale, $C_{N,s}$ est une constante positive de normalisation et vaut (e.g. [74, formula (A.1)])

$$C_{N,s} = \frac{4^s \Gamma(\frac{N}{2} + s)}{\pi^{N/2}} \cdot \frac{s}{\Gamma(1 - s)};$$

- comme un opérateur pseudo-différentiel [37, 50]

$$\mathcal{F} [(-\Delta)^s u](\xi) = |\xi|^{2s} \mathcal{F} u(\xi)$$

où \mathcal{F} dénote la transformée de Fourier ;

- comme un opérateur Dirichlet-to-Neumann [18]

$$\begin{aligned} \left\{ \begin{array}{ll} \operatorname{div} (t^{1-2s} U(x, t)) = 0 & x \in \mathbb{R}^N, t > 0 \\ U(x, 0) = u(x) & x \in \mathbb{R}^N \end{array} \right. & \rightsquigarrow \\ & \rightsquigarrow (-\Delta)^s u(x) = -c(N, s) \lim_{t \downarrow 0} t^{1-2s} \partial_t U(x, t) \end{aligned}$$

où $c(N, s)$ est une constante positive de normalisation ;

¹À noter que les opérateurs elliptiques, comme tout opérateur différentiel, sont purement locaux.

²Il est appelé aussi *Laplacien fractionnaire restreint* lorsque l'attention est fixée seulement sur les fonctions qui s'annulent en dehors d'un domaine fixé, voir par exemple [86].

³Ce n'est pas notre intention dans cet Résumé de fournir les hypothèses optimales de régularité où intégrabilité sur u pour que les définitions aient un sens : on travaille seulement formellement ici.

- comme le générateur infinitesimal du semigroupe de la chaleur $e^{t\Delta}$ dans \mathbb{R}^N subordonné au sens de Bochner avec fonction de Bernstein correspondante $\tau \mapsto \tau^s$ [76, Section 13.1]

$$(-\Delta)^s u(x) = \frac{s}{\Gamma(1-s)} \int_0^\infty \left(\frac{u(x) - e^{t\Delta} u(x)}{t} \right) \frac{dt}{t^s};$$

- comme le générateur infinitesimal d'un processus stochastique α -stable $(X_t)_{t \geq 0}$ où $\alpha = 2s$

$$(-\Delta)^s u(x) = \lim_{t \downarrow 0} \frac{u(x) - \mathbb{E}_x[u(X_t)]}{t}.$$

L'opérateur se révèle “nonlocal” au sens que pour le calculer (sur une fonction donnée et en un point fixé) ce n'est pas suffisant de connaître la fonction dans un voisinage du point ; inversement, une modification de la fonction loin du point concerne quoi qu'il en soit la valeur au point même (cette caractéristique est claire en utilisant la première définition) ; ceci n'est pas du tout le cas, par exemple, pour le Laplacien classique $-\Delta$. Les différentes, pourtant équivalentes, définitions pour $(-\Delta)^s$ permettent de prendre des approches différents lorsqu'on veut résoudre les problèmes associés, en engendrant une interaction fructueuse entre techniques variationnelles, la théorie des opérateurs pseudo-différentiels, l'analyse fonctionnelle et la théorie du potentiel.

Le Laplacien fractionnaire spectral. Le deuxième opérateur que l'on va étudier est un autre opérateur nonlocal dont la définition dépend fortement du domaine pris en considération. Fixons un domaine borné $\Omega \subset \mathbb{R}^N$, $s \in (0, 1)$ et dénotons $\delta(x) = \text{dist}(x, \partial\Omega)$. Pour une fonction mesurable⁴ $u : \Omega \rightarrow \mathbb{R}$ l'opérateur $(-\Delta|_\Omega)^s$ peut être défini :

- comme un opérateur intégral avec noyau singulier [82]

$$(-\Delta|_\Omega)^s u(x) = p.v. \int_\Omega [u(x) - u(y)] J(x, y) dy + \kappa(x) u(x), \quad x \in \Omega$$

où $J(x, y)$ et $\kappa(x)$ sont fonctions données par

$$\begin{aligned} J(x, y) &= \frac{s}{\Gamma(1-s)} \int_0^\infty \frac{p_\Omega(t, x, y)}{t^{1+s}} dt \quad \asymp \frac{\delta(x)\delta(y)}{|x-y|^{N+2s}} \cdot \frac{1}{|x-y|^2 + \delta(x)\delta(y)} \\ \kappa(x) &= \frac{s}{\Gamma(1-s)} \int_0^\infty \left(1 - \int_\Omega p_\Omega(t, x, y) \right) \frac{dt}{t^{1+s}} \quad \asymp \frac{1}{\delta(x)^{2s}} \end{aligned}$$

et $p_\Omega(t, x, y)$ est le noyau de la chaleur de Ω avec conditions homogènes ;

- via la décomposition spectrale de u : étant donnée $\{\varphi_j\}_{j \in \mathbb{N}}$ une base hilbertienne de $L^2(\Omega)$ constituée par les fonctions propres du Laplacien de Dirichlet $-\Delta|_\Omega$ associées à les valeurs propres λ_j , $j \in \mathbb{N}$,

$$(-\Delta|_\Omega)^s u(x) = \sum_{j=1}^\infty \lambda_j \widehat{u}_j \varphi_j(x), \quad \widehat{u}_j = \int_\Omega u_j \varphi_j;$$

- comme un opérateur Dirichlet-to-Neumann [16, 23]

$$\begin{aligned} \left\{ \begin{array}{ll} \text{div}(t^{1-2s} U(x, t)) = 0 & x \in \Omega, t > 0 \\ U(x, t) = 0 & x \in \partial\Omega, t > 0 \\ U(x, 0) = u(x) & x \in \Omega \end{array} \right. \rightsquigarrow \begin{array}{l} \\ \\ \end{array} \rightsquigarrow (-\Delta|_\Omega)^s u(x) = -c(N, s) \lim_{t \downarrow 0} t^{1-2s} \partial_t U(x, t) \end{aligned}$$

où $c(N, s)$ est une constante positive de normalisation ;

⁴Comme auparavant, ici on travaille seulement formellement.

- comme le générateur infinitesimal du semigroupe de la chaleur $e^{t\Delta|_\Omega}$ dans Ω subordonné au sens de Bochner avec fonction de Bernstein correspondante $\tau \mapsto \tau^s$ [76, Section 13.1] (et aussi [47])

$$(-\Delta|_\Omega)^s u(x) = \frac{s}{\Gamma(1-s)} \int_0^\infty \left(\frac{u(x) - e^{t\Delta|_\Omega} u(x)}{t} \right) \frac{dt}{t^s};$$

- comme le générateur infinitesimal d'un processus de Wiener arrêté subordonné $(\tilde{X}_t)_{t \geq 0}$

$$(-\Delta|_\Omega)^s u(x) = \lim_{t \downarrow 0} \frac{u(x) - \mathbb{E}_x[u(\tilde{X}_t)]}{t}.$$

À notre connaissance la définition via la décomposition spectrale est la plus utilisée dans un contexte d'EDP. Cependant elle cause restrictions sur le domaine de l'opérateur, et aucune théorie de conditions au bord non homogènes était disponible jusqu'à ce travail ; cela est une des contributions de cette thèse, voir Chapitre 7. Voilà pourquoi on va utiliser surtout la définition avec le noyau singulier ou celle obtenue par la subordination au sens de Bochner.

Interprétation stochastique et le nom des opérateurs. Comme on l'a déjà mentionné, les deux opérateurs ont un lien profond avec la théorie des processus stochastiques. En effet ils génèrent des processus de Lévy avec sauts, c'est-à-dire dont les trajectoires sont discontinues. L'opérateur $(-\Delta)^s$ génère un processus $(2s)$ -stable dans \mathbb{R}^N , ce qui signifie que sa distribution a une homogénéité d'ordre $2s$ dans le temps. Ce processus peut être construit aussi en subordonnant un mouvement Brownien dans le temps : étant donné un mouvement Brownien $(B_t)_{t \geq 0}$, au lieu de considérer un temps continu, on va l'évaluer dans des temps aléatoires $(T_t)_{t \geq 0}$ qui sont croissants presque sûrement. De cette façon on construit un nouveau processus $(B_{T_t})_{t \geq 0}$. Le temps aléatoire T_t n'est pas supposé avoir des trajectoires continues. Dans cette approche, on obtient finalement un processus de Lévy dont les sauts sont dus aux sauts temporels, comme si l'on était en train d'enregistrer le mouvement aléatoire d'une particule mais en arrêtant l'appareil d'une façon aléatoire, ensuite on le réallume et on repète (on pourrait aussi accélérer ou ralentir l'écoulement du temps). On rappelle que le générateur du mouvement Brownien est le Laplacien classique (négatif) $-\Delta$. Si le subordonateur est bien choisi, alors l'opération stochastique de la subordination a comme homologue analytique le calcul d'une puissance fractionnaire du générateur : c'est là qu'on voit apparaître $(-\Delta)^s$.

Le même type de construction serait valable aussi pour $(-\Delta|_\Omega)^s$, mais avec une différence importante. Avant de subordonner, on l'arrête lors de la première sortie de Ω . Pour cela on a dit que il génère un *mouvement Brownien arrêté subordonné*. Le mouvement Brownien arrêté est engendré par le Laplacien de Dirichlet $-\Delta|_\Omega$ et la subordination fournit sa puissance fractionnaire $(-\Delta|_\Omega)^s$. Ceci explique pourquoi on a choisi cette notation au lieu d'utiliser la plus commune A_s .

Nous précisons enfin que nous l'avons appelé *Laplacien fractionnaire spectral* pour nous conformer à comment il est appelé dans la bibliographie disponible. On pourrait aussi l'appeler *Laplacien de Dirichlet fractionnaire* et cela prendrait plus en compte la façon dont il est construit. Le mot *spectral* risque d'être trompeur, parce qu'il rappelle la définition via la décomposition spectrale qui a des restrictions assez naturelles à respecter et qui ne peut pas être appliquée à l'entière classe de fonctions qu'on va considérer ici. Dans le même esprit on pourrait aussi appeler *Laplacien fractionnaire de Dirichlet* l'opérateur obtenu avec une restriction du Laplacien fractionnaire à les fonctions avec condition au bord nulles. En effet, cet opérateur génère un *mouvement Brownien subordonné arrêté*, c'est-à-dire le processus obtenu en arrêtant le processus de Lévy $(2s)$ -stable lors de la première sortie du domaine fixé. On remarque que "subordonné arrêté" et "arrêté subordonné" sont différents en nature et que de même "fractionnaire de Dirichlet" et "de Dirichlet fractionnaire" ne sont pas la même chose.

Solutions avec explosion au bord. Dans la théorie des équations elliptiques semilinéaires, une fonction qui résout

$$-\Delta u + f(u) = 0 \quad \text{dans } \Omega \subset \mathbb{R}^N \text{ ouvert et borné,}$$

avec condition au bord

$$\lim_{x \rightarrow \partial\Omega} u(x) = +\infty,$$

est appelée *solution avec explosion au bord* où aussi *grande solution*. On a énormément de bibliographie disponible sur ce sujet qui remonte à Bieberbach [10], pour $N = 2$ et $f(u) = e^u$. Keller [56] et Osserman [70] indépendamment ont établi une condition nécessaire et suffisante sur la nonlinéarité f pour l'existence d'une solution avec explosion au bord : elle est

$$\int^{+\infty} \frac{dt}{\sqrt{F(t)}}, \quad \text{où } F' = f \geq 0$$

et elle est connue comme *condition de Keller-Osserman*. On peut trouver ces solutions avec singularité au bord dans nombreuses applications : par exemple, Loewner et Nirenberg [63] ont étudié le cas $f(u) = u^{(N+2)/(N-2)}$, $N \geq 3$ qui est strictement lié au *problème de Yamabe singulier* en géométrie conforme, tandis que Labutin [59] a complètement caractérisé la classe des ensembles Ω qui admettent une grande solution pour $f(u) = u^q$, $q > 1$, à travers des méthodes de capacité inspirés par la théorie des *spatial branching processes*, qui sont des processus stochastiques particuliers ; voir aussi les travaux purement probabilistes de Le Gall [61] et Dhersin, Le Gall [35] traitants le cas $q = 2$ et le cas délicat des domaines non lisses. Mselati [68] a complètement classifié les solutions positives en fonction de la trace au bord, qui peut être $+\infty$ sur une partie du bord et une mesure qui ne se concentre pas sur des ensemble de capacité zéro ; voir aussi Dumont, Dupaigne, Goubet, Rădulescu [40] pour le cas d'une nonlinéarité oscillante. On se réfère au très récent livre de Marcus et Véron [65, Chapitres 5 et 6] pour d'autres références.

Fonctions s -harmoniques grandes, formules d'intégration par partie et théorie faible L^1 . Dans le contexte fractionnaire, notre point de départ est que des solutions avec explosion au bord apparaissent même dans des problèmes linéaires. En particulier, il est possible de donner des *fonctions s -harmoniques grandes*, c'est-à-dire des fonctions qui résolvent

$$\left\{ \begin{array}{ll} (-\Delta)^s u = 0 & \text{dans } \Omega \\ u = 0 & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} u(x) = +\infty \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} (-\Delta|_{\Omega})^s u = 0 & \text{dans } \Omega \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} u(x) = +\infty \end{array} \right.$$

voir Lemmata 2.11, 7.10 and (7.20) ci-dessous.

Ces exemples montrent que d'un coté les problèmes pour $(-\Delta)^s$ sont mal posés au sens de Hadamard si seulement les valeurs à l'extérieur de Ω sont données et différentes conditions au bord doivent être prises en consideration, tandis qu'ils montrent le besoin de conditions au bord non nulles pour $(-\Delta|_{\Omega})^s$. Cela veut dire aussi qu'on a besoin des notions de solution faible qui puissent traiter ces nouvelles conditions au bord. On réfère à Grubb [50, Théorèmes 4 et 5] pour des résultats très récents pour la théorie de régularité pour les problèmes de ce type impliquants le Laplacien fractionnaire, et on cite Dhifli, Mâagli et Zribi [36] où un premier essai a été fait sur le Laplacien fractionnaire spectral. On répond à ces questions dans la Section 2.2 et le Théorème 6.5 : la trace est une limite au bord pondérée avec puissances de la distance au bord, $\delta(x)^{1-s}u(x)$ dans le cas du Laplacien fractionnaire et $\delta(x)^{2-2s}u(x)$ dans celui du Laplacien fractionnaire spectral.

Dans les deux cas on a eu besoin de développer une théorie faible L^1 de type Stampacchia avec des mesures comme données, capable de traiter aussi cette nouvelle classe de fonctions s -harmoniques. Solutions de⁵

$$\left\{ \begin{array}{ll} (-\Delta)^s u = \lambda & \text{dans } \Omega \\ u = \mu & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ \frac{u}{w_1} = \nu & \text{sur } \partial\Omega \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} (-\Delta|_{\Omega})^s u = \xi & \text{dans } \Omega \\ \frac{u}{w_2} = \zeta & \text{sur } \partial\Omega \end{array} \right.$$

⁵Ici w_1 et w_2 vont dénoter des fonctions de référence définies dans Ω dont le comportement est comparable respectivement à ceux de δ^{s-1} et δ^{2s-2} : il faut interpreter la condition au bord comme une limite lorsqu'on approche un point du bord.

où $\lambda, \xi \in \mathcal{M}(\Omega)$, $\mu \in \mathcal{M}(\mathbb{R}^N \setminus \overline{\Omega})$, $\nu, \zeta \in \mathcal{M}(\partial\Omega)$ sont des mesures de Radon qui satisfont certaines hypothèses, vont ressembler à

$$\begin{aligned} \int_{\Omega} u(-\Delta)^s \phi &= \int_{\Omega} \phi d\lambda - \int_{\mathbb{R}^N \setminus \Omega} (-\Delta)^s \phi d\mu + \int_{\partial\Omega} \frac{\phi}{\delta^s} \Big|_{\partial\Omega} d\nu & u \in L^1(\Omega), \chi_{\Omega}(-\Delta)^s \phi \in C_c^{\infty}(\Omega), \\ \int_{\Omega} u(-\Delta|_{\Omega})^s \phi &= \int_{\Omega} \phi d\xi - \int_{\partial\Omega} \frac{\partial \phi}{\partial \nu} d\zeta & u \in L^1(\Omega, \delta(x)dx), (-\Delta|_{\Omega})^s \phi \in C_c^{\infty}(\Omega). \end{aligned}$$

Elles seront présentées dans les Sections 2.3 et 7.4. Ces définitions sont inspirées de et, d'une certaine manière, généralisent des formules d'intégration par parties. Les techniques sont basées surtout sur l'analyse du potentiel, et exploitent la représentation via la fonction de Green, les noyaux de Poisson et de Martin, dont la construction est incluse. Une présentation plus précise des résultats et l'énoncé rigoureux des découvertes principales se trouvent dans les Chapitres 1 et 6.

Problèmes nonlinéaires. Jusque là on a vu que des singularités au bord peuvent apparaître même en présence de donnée assez régulières. Lorsqu'on s'occupe d'un problème nonlinéaire, cette perte de régularité au bord donne lieu à une grande classe de solutions avec une singularité *prescrite* au bord, présentée dans le Chapitre 3 et la Section 8.1. On va étudier les problèmes du type

$$\left\{ \begin{array}{ll} (-\Delta)^s u = f(x, u) & \text{dans } \Omega \\ u = g & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ \frac{u}{w_1} = h & \text{sur } \partial\Omega \end{array} \right. \quad \text{et} \quad \left\{ \begin{array}{ll} (-\Delta|_{\Omega})^s u = f(x, u) & \text{dans } \Omega \\ \frac{u}{w_2} = h & \text{sur } \partial\Omega. \end{array} \right.$$

Dans ces cas là, les fonctions s -harmoniques grandes peuvent être utilisées comme sous- et sursolution si le signe de la nonlinéarité le permet. Pour cela, on a résolu ces problèmes nonlinéaires en adaptant les techniques de Clément et Sweers [28] et Montenegro et Ponce [67].

Si l'on prend par exemple la donnée $h \equiv 1$, elle va nous donner une solution avec une explosion au bord d'ordre δ^{s-1} et δ^{2s-2} . Pourtant cette solution ne sera pas *grande*, dans le sens suivant. Dans la théorie classique du Laplacien, les grandes solutions sont engendrées par une compensation entre la condition au bord - qui essaie de relever la solution à l'infini - et la nonlinéarité - qui la pousse vers le bas, en raison du signe négatif. La nonlinéarité doit fournir une réponse assez forte à la condition au bord, et dans ce sens là la condition de Keller-Osserman quantifie le niveau nécessaire pour que ce phénomène ait lieu. Mais pour l'instant, dans le contexte fractionnaire, on a seulement présenté des solutions dont la singularité au bord est due à des caractéristiques linéaires des opérateurs. Alors, encore une question attend une réponse : est-il possible de trouver des solutions dont la singularité au bord soit engendrée non pas par les caractéristiques de l'opérateur, mais plutôt par un phénomène de compensation ?

Grandes solutions pour le Laplacien fractionnaire. On commence à répondre dans les Chapitres 4 et 8, en donnant des conditions suffisantes pour l'existence d'une solution à $(-\Delta)^s u + f(u) = 0$ et $(-\Delta|_{\Omega})^s u + f(u) = 0$ avec une singularité au bord qui soit d'ordre supérieur à ceux décrits dans le paragraphe précédent, de sorte que les conditions au bord seront

$$\frac{u}{w_1} = +\infty \text{ sur } \partial\Omega \quad \text{et} \quad \frac{u}{w_2} = +\infty \text{ sur } \partial\Omega$$

qui sont équivalentes à

$$\delta^{1-s} u|_{\partial\Omega} = +\infty \quad \text{et} \quad \delta^{2-2s} u|_{\partial\Omega} = +\infty.$$

Le cas de la nonlinéarité de type puissance $f(u) = u^p$ pour $(-\Delta)^s$ a été étudié par Felmer et Quaas [43] et Chen, Felmer et Quaas [25]. On va étendre ces résultats en montrant des bornes explicites sur p , qui sont

$$p \in \left(1 + 2s, 1 + \frac{2s}{1-s} \right),$$

en admettant des nonlinéarités plus générales et un endroit plus clair où travailler, surtout lorsqu'on parle de condition au bord. On démontre aussi l'estimation ponctuelle suivante, qui est valable dans le domaine entier :

$$F' = f, \quad \int_{u(x)}^{\infty} \frac{dt}{\sqrt{F(t)}} \geq c \delta(x)^s;$$

cette borne supérieure vient d'un calcul explicite d'une sursolution pour le problème. Dans le cas particulier où $f(u) = u^p$ elle devient

$$u \leq C \delta^{-2s/(p-1)}.$$

Dans le cas du Laplacien fractionnaire spectral, on va étudier les nonlinéarités de type puissance. On regarde

$$\begin{cases} (-\Delta|_{\Omega})^s u = -u^p & \text{dans } \Omega \\ \delta^{2-2s} u = +\infty & \text{sur } \partial\Omega, \end{cases}$$

et on prouve l'existence d'une solution $u \in L^1(\Omega, \delta(x)dx)$ pour

$$p \in \left(1 + s, 1 + \frac{s}{1-s}\right).$$

Là aussi on a une estimation ponctuelle, qui est dérivée d'une sursolution, et qui est

$$u \leq C \delta^{-2s/(p-1)} \text{ dans } \Omega.$$

Cette croissance coïncide avec celle du Laplacien classique et elle est optimale pour $s = 1$.

Lorsqu'on s'occupe de $f(u) = u^p$ pour $(-\Delta)^s$ ou $(-\Delta|_{\Omega})^s$, une question naturelle est de savoir si les gammes des exposants admissibles convergent asymptotiquement lorsque $s \uparrow 1$ vers la gamme des exposants admissibles pour $-\Delta$, qui est donnée par la condition de Keller-Osserman et qui est $p \in (1, +\infty)$. La réponse est *non*, clairement. Si d'un côté on a

$$1 + \frac{2s}{1-s} \rightarrow +\infty \quad \text{et} \quad 1 + \frac{s}{1-s} \rightarrow +\infty \quad \text{lorsque } s \uparrow 1,$$

de l'autre on a seulement que

$$1 + 2s \rightarrow 3 \quad \text{et} \quad 1 + s \rightarrow 2 \quad \text{lorsque } s \uparrow 1.$$

Pourtant cela n'est pas décourageant. Dans les deux contextes fractionnaires on a besoin respectivement de $u \in L^1(\Omega)$ et $u \in L^1(\Omega, \delta(x)dx)$ pour donner un sens aux opérateurs. Celle-là est une hypothèse que l'on n'a pas dans le problème classique, et il est donc raisonnable d'obtenir des gammes plus petites pour p . De plus, on sait que la solution classique se comporte comme

$$u \asymp \delta^{-2/(p-1)}$$

et qu'elle est dans $L^1(\Omega)$ si $p > 3$, alors qu'elle est dans $L^1(\Omega, \delta(x)dx)$ si $p > 2$; ainsi, en ce sens là, on a la convergence asymptotique des gammes des exposants admissibles.

Courbures directionelles nonlocales. Dans le dernier chapitre du manuscrit on va étudier une notion fractionnaire de courbure directionelle pour des variétés lisses de dimension $N - 1$ dans \mathbb{R}^N . Cet intérêt est motivé par le contexte fractionnaire dans lequel beaucoup des problèmes classiques ont été reformulés. En particulier, en géométrie différentielle et en analyse géométrique, beaucoup de nouveaux résultats ont été obtenus à regard de la *diffusion par courbure moyenne* et le problème lié des *surfaces minimales* dans un contexte nonlocal.

Comme l'on va voir, le concept de courbure moyenne nonlocale peut être naturellement associé à la moyenne des courbures directionelles appropriées, qui asymptotiquement convergent à leur homologues classiques. Par ailleurs, les courbures nonlocales semblent avoir un comportement plus désordonné et

fantaisiste de celles classiques. En particulier, les courbures directionnelles minimale et maximale ne sont pas atteintes, en général, le long de deux directions orthogonales et les ensembles où elles sont atteintes peut même être prescrit d'une façon arbitraire⁶. Des raisons pour introduire ces objets sont données dans le Paragraphe 9.2.2.

PARTIE I : GRANDES SOLUTIONS POUR $(-\Delta)^s$

Dans la première Partie on va étudier une notion de solution faible à des problèmes séminéaires pour le Laplacien fractionnaire $(-\Delta)^s$, c'est-à-dire l'opérateur integral défini par (voir par exemple Di Nezza, Palatucci et Valdinoci [37] pour une introduction)

$$(-\Delta)^s f(x) = C_{N,s} PV \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x - y|^{N+2s}} dy,$$

où $C_{N,s}$ est la constante positive de normalisation définie dans (3).

Pour pouvoir faire cela, on aura besoin de développer une théorie pour le problème de Dirichlet avec le Laplacien fractionnaire et des mesures comme données (voir Karlsen, Petitta et Ulusoy [55] et Chen et Véron [26] pour des résultats récents en cette direction). On va se concentrer sur les solutions avec un comportement explosif au bord du domaine prescrit, connues dans la littérature comme *grandes solutions* ou *solutions avec explosion au bord*.

On rappelle que dans le cas classique (voir [5, Théorème 6.9]), à toute mesure non négative de Borel $\mu \in \mathcal{M}(\partial B)$ il est possible d'associer, en utilisant le noyau de Poisson, une fonction harmonique dans B qui a μ pour trace au bord. Inversement, toute fonction harmonique sur la boule B admet une trace sur ∂B qui est une mesure de Borel non négative (voir [5, Théorème 6.15]).

Lorsqu'on étudie le problème séminéaire pour le Laplacien, les solutions peuvent atteindre la valeur $+\infty$ comme donnée au bord. Plus précisément, si Ω est un domaine lisse borné et ouvert, et si f est croissante avec $f(0) = 0$, d'après les travaux de Keller [56] et Osserman [70], on obtient que l'équation

$$\begin{cases} \Delta u = f(u) & \text{dans } \Omega \subseteq \mathbb{R}^N \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} u(x) = +\infty \end{cases}$$

possède une solution si et seulement si f satisfait la *condition de Keller-Osserman* (déjà nommée en haut).

Dans le contexte fractionnaire, notre point de départ est que l'on peut fournir des fonctions telles que

$$\begin{cases} (-\Delta)^s u = 0 & \text{dans une région bornée et ouverte } \Omega \subseteq \mathbb{R}^N, \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} u(x) = +\infty. \end{cases}$$

Par exemple une fonction s -harmonique grande sur la boule B est

$$u_\sigma(x) = \begin{cases} \frac{c(N,s)}{(1-|x|^2)^\sigma} & \text{in } B \\ \frac{c(N,s+\sigma)}{(|x|^2-1)^\sigma} & \text{in } \mathbb{R}^N \setminus \overline{B} \end{cases} \quad \sigma \in (0, 1-s), \quad c(N,s) = \frac{\Gamma(N/2) \sin(\pi s)}{\pi^{1+N/2}}.$$

En plus, lorsque $\sigma \rightarrow 1-s$ on retrouve l'exemple suivant donné par Bogdan, Byczkowski, Kulczycki, Ryznar, Song, et Vondraček [13], qui est qualitativement différent du précédent :

$$u_{1-s}(x) = \begin{cases} \frac{c(N,s)}{(1-|x|^2)^{1-s}} & \text{in } B, \\ 0 & \text{in } \mathbb{R}^N \setminus \overline{B}. \end{cases}$$

⁶On peut trouver l'énoncé rigoureux de ces résultats dans le Paragraphe 9.2.3.

La fonction u_{1-s} définie ainsi satisfait

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B \\ u = 0 & \text{in } \mathbb{R}^N \setminus \overline{B} \end{cases}$$

et montre comme les problèmes où seulement les valeurs extérieures sont prescrits sont mal posés en sens classique. Différentes conditions au bord doivent être considérées : en effet, dans le premier cas on a une fonction s -harmonique associée aux valeurs prescrites de u_σ à l'extérieur de B ; dans le deuxième cas toute la masse de la donnée se concentre sur ∂B . Cela veut dire que l'on a besoin d'une notion de solution faible qui puisse affronter en même temps ces deux données au bord, une sur le complémentaire du domaine et une sur le bord. On renvoi à Grubb [50, Théorème 4 et 5] pour des résultats très récents sur la régularité du Laplacien fractionnaire traitant ces types de problèmes.

La notion de fonction s -harmoniques. On commence par la définition de fonction s -harmonique, en suivant Landkof [60, §6.20]. On dénote

$$\eta(x) = \begin{cases} \frac{c(N, s)}{|x|^N (|x|^2 - 1)^s} & |x| > 1 \\ 0 & |x| \leq 1 \end{cases}$$

et

$$\eta_r(x) = \frac{1}{r^N} \eta\left(\frac{x}{r}\right) = \begin{cases} \frac{c(N, s) r^{2s}}{|x|^N (|x|^2 - r^2)^s} & |x| > r, \\ 0 & |x| \leq r. \end{cases}$$

La constante $c(N, s)$ est choisie d'une façon telle que

$$\int_{\mathbb{R}^N} \eta(x) dx = \int_{\mathbb{R}^N} \eta_r(x) dx = 1$$

et en conséquence, voir [60, §6.19],

$$c(N, s) = \frac{\Gamma(N/2) \sin(\pi s)}{\pi^{1+N/2}}.$$

La définition de fonction s -harmonique est donnée par

DÉFINITION | On dit que une fonction mesurable $u : \mathbb{R}^N \rightarrow [0, +\infty]$ est s -harmonique sur un ensemble ouvert $\Omega \subseteq \mathbb{R}^N$ si $u \in C(\Omega)$ et pour tout $x \in \Omega$ et $0 < r < \text{dist}(x, \partial\Omega)$

$$u(x) = \int_{CB_r(x)} \frac{c(N, s) r^{2s}}{|y - x|^N (|y - x|^2 - r^2)^s} u(y) dy = (\eta_r * u)(x).$$

Une formule d'intégration par parties

Pour toute couple de fonctions u, v dans la classe de Schwartz \mathcal{S} , le fait que $(-\Delta)^s$ soit auto-adjoint implique que

$$\int_{\mathbb{R}^N} u (-\Delta)^s v = \int_{\mathbb{R}^N} (-\Delta)^s u v,$$

et cela suit de la représentation du Laplacien fractionnaire via la transformée de Fourier, voir [37, Paragraphe 3.1]. Si l'on coupe en deux morceaux l'espace \mathbb{R}^N on obtient

$$\int_{\Omega} u (-\Delta)^s v - \int_{\Omega} (-\Delta)^s u v = \int_{\mathbb{R}^N \setminus \Omega} (-\Delta)^s u v - \int_{\mathbb{R}^N \setminus \Omega} u (-\Delta)^s v.$$

On dénote par

$$C^{2s+\varepsilon}(\Omega) = \{v : \mathbb{R}^N \rightarrow \mathbb{R} \text{ telle que } v \in C(\Omega) \text{ et pour chaque } K \subset\subset \Omega$$

$$\text{il existe } \alpha = \alpha(K, v) \text{ tel que } v \in C^{2s+\alpha}(K)\}.$$

PROPOSITION | Soit $\Omega \subseteq \mathbb{R}^N$ ouvert et borné. Si $u \in C^{2s+\varepsilon}(\Omega) \cap L^\infty(\mathbb{R}^N)$ et $v = 0$ in $\mathbb{R}^N \setminus \Omega$, $v \in C^{2s+\varepsilon}(\Omega) \cap C^s(\mathbb{R}^N)$ et $(-\Delta)^s v \in L^1(\mathbb{R}^N)$, alors

$$\int_{\Omega} u (-\Delta)^s v - \int_{\Omega} (-\Delta)^s u v = - \int_{\mathbb{R}^N \setminus \Omega} u (-\Delta)^s v.$$

La preuve se trouve dans la Section 2.2.1.

À partir de maintenant l'ensemble $\Omega \subseteq \mathbb{R}^N$ sera un domaine ouvert et borné avec bord $C^{1,1}$. On va montrer dans la Section 2.2

PROPOSITION | Soit $\delta(x) = \text{dist}(x, \partial\Omega)$ pour tout $x \in \mathbb{R}^N$, et $u \in C^{2s+\varepsilon}(\Omega)$ telle que

$$\delta(x)^{1-s} u(x) \in C(\overline{\Omega}) \quad \text{et} \quad \int_{C\Omega} \frac{|u(x)|}{1+|x|^{N+2s}} \cdot \frac{dx}{\delta(x)^s} < +\infty.$$

Soit

a) $G_{\Omega}(x, y)$, $x, y \in \Omega$, $x \neq y$, la fonction de Green du Laplacien fractionnaire sur Ω , c'est-à-dire

$$G_{\Omega}(x, y) = \frac{-C_{N,-s}}{|x-y|^{N-2s}} - H_{\Omega}(x, y)$$

et H_{Ω} la seule fonction dans $C^s(\mathbb{R}^N)$ qui résout

$$\begin{cases} (-\Delta)^s H_{\Omega}(x, \cdot) = 0 & \text{in } \Omega \\ H_{\Omega}(x, y) = \frac{-C_{N,-s}}{|x-y|^{N-2s}} & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \end{cases}$$

ponctuellement,⁷

b) $P_{\Omega}(x, y) = -(-\Delta)^s G_{\Omega}(x, y)$, $x \in \Omega, y \in \mathbb{R}^N \setminus \overline{\Omega}$, le noyau de Green correspondant,

c) pour $x \in \Omega, \theta \in \partial\Omega$,⁸

$$M_{\Omega}(x, \theta) = \lim_{\substack{y \rightarrow \theta \\ y \in \Omega}} \frac{G_{\Omega}(x, y)}{\delta(y)^s},$$

d) pour $\theta \in \partial\Omega$,

$$Eu(\theta) = \lim_{\substack{x \rightarrow \theta \\ x \in \Omega}} \frac{u(x)}{\int_{\partial\Omega} M_{\Omega}(x, \theta') d\mathcal{H}(\theta')},$$

où la limite est bien définie grâce au Lemme 2.7 en bas.

Alors on a la formule d'intégration par parties

$$\int_{\Omega} u (-\Delta)^s v = \int_{\Omega} (-\Delta)^s u v - \int_{\mathbb{R}^N \setminus \overline{\Omega}} u (-\Delta)^s v + \int_{\partial\Omega} Eu(\theta) D_s v(\theta) d\mathcal{H}(\theta)$$

où

$$D_s v(\theta) = \lim_{\substack{x \rightarrow \theta \\ x \in \Omega}} \frac{v(x)}{\delta(x)^s}, \quad \theta \in \partial\Omega$$

pour toute $v \in C^s(\mathbb{R}^N)$ telle que $(-\Delta)^s v|_{\Omega} \in C_c^\infty(\Omega)$ et $v \equiv 0$ dans $\mathbb{R}^N \setminus \Omega$. Une telle limite existe et elle est continue en θ grâce au Lemme 2.12. En plus, on a la formule de représentation

$$u(x) = \int_{\Omega} G_{\Omega}(x, y) (-\Delta)^s u(y) dy - \int_{\mathbb{R}^N \setminus \overline{\Omega}} (-\Delta)^s G_{\Omega}(x, y) u(y) dy + \int_{\partial\Omega} D_s G_{\Omega}(x, \theta) Eu(\theta) d\mathcal{H}(\theta).$$

⁷ La construction de H peut être trouvée dans la preuve du Théorème 2.1.

⁸ Ceci est un noyau de Martin de Ω réadapté.

Le problème de Dirichlet

Étant fixé $x \in \Omega$, le noyau de Poisson satisfait

$$0 \leq -(-\Delta)^s G_\Omega(x, y) \leq c \min \left\{ \delta(y)^{-s}, \delta(y)^{-N-2s} \right\}, \quad y \in \mathbb{R}^N \setminus \overline{\Omega}$$

pour quelque constante $c > 0$ indépendante de y , en conséquence de (2.11) et (2.19) en bas. En particulier, toute condition de Dirichlet $u = g$ dans $\mathbb{R}^N \setminus \overline{\Omega}$ qui satisfait (1) en bas est admissible dans la formule de représentation. On démontre le suivant

THÉORÈME | Soit $f \in C^\alpha(\overline{\Omega})$ pour quelque $\alpha \in (0, 1)$, $g : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow \mathbb{R}$ une fonction mesurable qui satisfait

$$\int_{\mathbb{R}^N \setminus \overline{\Omega}} |g(y)| \min \left\{ \delta(y)^{-s}, \delta(y)^{-N-2s} \right\} dy < +\infty \quad (1)$$

et $h \in C(\partial\Omega)$. Alors la fonction définie par

$$u(x) = \begin{cases} \int_{\Omega} f(y) G_\Omega(y, x) dy - \int_{\mathbb{R}^N \setminus \overline{\Omega}} g(y) (-\Delta)^s G_\Omega(x, y) dy + \\ \quad + \int_{\partial\Omega} h(\theta) M_\Omega(x, \theta) d\mathcal{H}(\theta), & x \in \Omega \\ g(x), & x \in \mathbb{R}^N \setminus \overline{\Omega} \end{cases}$$

appartient à $C^{2s+\varepsilon}(\Omega)$, satisfait (1.8), et elle est la seule solution ponctuelle de

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \\ Eu = h & \text{on } \partial\Omega. \end{cases}$$

En plus, si $g \in C(\overline{V_\varepsilon})$ pour quelque $\varepsilon > 0$, où $V_\varepsilon = \{x \in \mathbb{R}^N \setminus \overline{\Omega} : \delta(x) < \varepsilon\}$ et $h = 0$, alors $u \in C(\overline{\Omega})$.

REMARQUE | Même si ce n'est pas important d'écrire $\int_{\mathbb{R}^N \setminus \Omega}$ ou $\int_{\mathbb{R}^N \setminus \overline{\Omega}}$, par la deuxième notation on veut souligner que le bord $\partial\Omega$ a une importance non triviale en cet endroit.

REMARQUE | Vu que $\partial\Omega \in C^{1,1}$, on va exploiter le comportement de la fonction de Green et du noyau de Poisson décrit par [27, Théorème 2.10 et équation (2.13) resp.] : ils existent $c_1 = c_1(\Omega, s) > 0$, $c_2 = c_2(\Omega, s)$ tels que

$$\begin{aligned} \frac{\delta(x)^s}{c_1 \delta(y)^s (1 + \delta(y))^s |x - y|^N} &\leq -(-\Delta)^s G_\Omega(x, y) \leq \\ &\leq \frac{c_1 \delta(x)^s}{\delta(y)^s (1 + \delta(y))^s |x - y|^N}, \end{aligned} \quad x \in \Omega, y \in \mathbb{R}^N \setminus \overline{\Omega},$$

et

$$\begin{aligned} \frac{1}{c_2 |x - y|^N} (|x - y|^2 \wedge \delta(x) \delta(y))^s &\leq G_\Omega(x, y) \leq \\ &\leq \frac{c_2}{|x - y|^N} (|x - y|^2 \wedge \delta(x) \delta(y))^s \end{aligned} \quad x, y \in \Omega.$$

REMARQUE: CONSTRUCTION DE FONCTIONS s -HARMONIQUES GRANDES | Le cas où $f = 0$ dans le Théorème au dessus correspond aux fonctions s -harmoniques : lorsque $h \not\equiv 0$ on a que u automatiquement est singulière quelque part au bord (par définition de E), tandis que si $h \equiv 0$ des fonctions s -harmoniques grandes peuvent être construites comme suit. On prends une g positive quelconque avec

$$\lim_{\delta(x) \downarrow 0} g(x) = +\infty$$

et soit $g_n = \min\{g, n\}$, $n \in \mathbb{N}$. D'après le Théorème au dessus, les solutions correspondantes $u_n \in C(\overline{\Omega})$. En particulier, $u_n = n$ sur $\partial\Omega$ et par le Principe du Maximum $\{u_n\}_{n \in \mathbb{N}}$ est croissante. Alors

$$u_n(x) = - \int_{\mathbb{R}^N \setminus \Omega} g_n(y) (-\Delta)^s G_\Omega(x, y) dy$$

augmente jusqu'à une fonction s -harmonique u telle que

$$\liminf_{\substack{x \in \Omega \\ x \rightarrow \partial\Omega}} u(x) \geq \liminf_{\substack{x \in \Omega \\ x \rightarrow \partial\Omega}} u_n(x) = n, \quad \text{pour tout } n \in \mathbb{N}.$$

Ensuite, on a le droit d'introduire l'espace des fonctions test

$$\mathcal{T}(\Omega) = \{\phi \in C^\infty(\Omega) : \text{il existe } \psi \in C_c^\infty(\Omega) \text{ tel que } (-\Delta)^s \phi = \psi \text{ dans } \Omega, \phi = 0 \text{ dans } \mathbb{R}^N \setminus \overline{\Omega}, E\phi = 0 \text{ sur } \partial\Omega\}.$$

À noter que $\mathcal{T}(\Omega) \subseteq C^s(\mathbb{R}^N)$, grâce à [73, Proposition 1.1]. Inspirée par la formule d'intégration par parties, on donne la suivante définition de solution faible

DÉFINITION | *Étant donnée trois mesures de Radon $\lambda \in \mathcal{M}(\Omega)$, $\mu \in \mathcal{M}(\mathbb{R}^N \setminus \Omega)$ et $\nu \in \mathcal{M}(\partial\Omega)$, telles que*

$$\int_{\Omega} \delta(x)^s d|\lambda|(x) < +\infty, \quad \int_{\mathbb{R}^N \setminus \overline{\Omega}} \min\{\delta(x)^{-s}, \delta(x)^{-N-2s}\} d|\mu|(x) < +\infty, \quad |\nu|(\Omega) < +\infty$$

on dit que une fonction $u \in L^1(\Omega)$ est une solution de

$$\begin{cases} (-\Delta)^s u = \lambda & \text{dans } \Omega \\ u = \mu & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = \nu & \text{sur } \partial\Omega \end{cases}$$

si pour tout $\phi \in \mathcal{T}(\Omega)$

$$\int_{\Omega} u(x) (-\Delta)^s \phi(x) dx = \int_{\Omega} \phi(x) d\lambda(x) - \int_{\mathbb{R}^N \setminus \overline{\Omega}} (-\Delta)^s \phi(x) d\mu(x) + \int_{\partial\Omega} D_s \phi(\theta) d\nu(\theta).$$

Les intégrales dans la définition sont tous bien définies d'après le Lemme 2.12. Grâce à la formule d'intégration par partie, on montre

THÉORÈME | *Étant donnée deux mesures de Radon $\lambda \in \mathcal{M}(\Omega)$ et $\mu \in \mathcal{M}(\mathbb{R}^N \setminus \overline{\Omega})$ telles que*

$$\int_{\Omega} G_\Omega(x, y) d|\lambda|(y) < +\infty, \quad \text{and} \quad - \int_{\mathbb{R}^N \setminus \overline{\Omega}} (-\Delta)^s G_\Omega(x, y) d|\mu|(y) < +\infty,$$

pour presque tout $x \in \Omega$, et une mesure de Radon $\nu \in \mathcal{M}(\partial\Omega)$ telle que $|\nu|(\partial\Omega) < +\infty$, le problème

$$\begin{cases} (-\Delta)^s u = \lambda & \text{dans } \Omega \\ u = \mu & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = \nu & \text{sur } \partial\Omega \end{cases}$$

admet une solution unique $u \in L^1(\Omega)$ en un sens faible. De plus, on a la formule de représentation

$$u(x) = \int_{\Omega} G_\Omega(x, y) d\lambda(y) - \int_{\mathbb{R}^N \setminus \overline{\Omega}} (-\Delta)^s G_\Omega(x, y) d\mu(y) + \int_{\partial\Omega} M_\Omega(x, y) d\nu(y)$$

et

$$\|u\|_{L^1(\Omega)} \leq C \left(\|\delta(x)^s \lambda\|_{\mathcal{M}(\Omega)} + \|\min\{\delta(x)^{-s}, \delta(x)^{-N-2s}\} \mu\|_{\mathcal{M}(\mathbb{R}^N \setminus \overline{\Omega})} + \|\nu\|_{\mathcal{M}(\partial\Omega)} \right)$$

pour quelque constante $C = C(n, s, \Omega) > 0$.

On va conclure la Section 2.2 avec

PROPOSITION | *La solution faible de*

$$\begin{cases} (-\Delta)^s u(x) = \frac{1}{\delta(x)^\beta} & \text{dans } \Omega \\ u = 0 & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = 0 & \text{sur } \partial\Omega \end{cases}$$

satisfait

$$\begin{aligned} c_1 \delta(x)^s &\leq u(x) \leq c_2 \delta(x)^s && \text{pour } 0 < \beta < s, \\ c_3 \delta(x)^s \log \frac{1}{\delta(x)} &\leq u(x) \leq c_4 \delta(x)^s \log \frac{1}{\delta(x)} && \text{pour } \beta = s, \\ c_5 \delta(x)^{-\beta+2s} &\leq u(x) \leq c_6 \delta(x)^{-\beta+2s} && \text{pour } s < \beta < 1 + s. \end{aligned}$$

En plus, ils existent deux constantes $\underline{c} = \underline{c}(n, s, \Omega) > 0$ et $\overline{c} = \overline{c}(n, s, \Omega)$ telles que la solution de

$$\begin{cases} (-\Delta)^s u = 0 & \text{dans } \Omega \\ u = g & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = 0 & \text{dans } \partial\Omega \end{cases}$$

satisfait

$$\underline{c} \underline{g}(\delta(x)) := \underline{c} \inf_{\delta(y)=\delta(x)} g(y) \leq u(x) \leq \overline{c} \sup_{\delta(y)=\delta(x)} g(y) =: \overline{c} \overline{g}(\delta(x)) \quad x \in \Omega,$$

pour toute g admissible et telle que $\underline{g}, \overline{g}$ sont des fonctions croissantes en 0^+ et

$$\lim_{t \downarrow 0} \underline{g}(t) = +\infty.$$

Le problème de Dirichlet semilinéaire

On va considérer $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ sous les hypothèses

f.1) $f \in C(\Omega \times \mathbb{R})$, $f \in L^\infty(\Omega \times I)$ pour chaque $I \subseteq \mathbb{R}$ borné

f.2) $f(x, 0) = 0$ pour tout $x \in \Omega$, et $f(x, t) \geq 0$ pour tout $x \in \Omega$, $t > 0$,

et tous les données au bord g admissibles.

Une fois que l'on a construit les fonctions s -harmoniques grandes, on démontre tout d'abord le résultat suivant.

THÉORÈME | *Soit $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ une fonction qui satisfait f.1). Soit $g : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow \mathbb{R}$ une fonction mesurable bornée. On suppose que le problème nonlinéaire*

$$\begin{cases} (-\Delta)^s u = -f(x, u) & \text{dans } \Omega \\ u = g & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = 0 & \text{sur } \partial\Omega \end{cases}$$

admet une soussolution $\underline{u} \in L^1(\Omega)$ et une sursolution $\overline{u} \in L^1(\Omega)$ au sense faible

$$\begin{cases} (-\Delta)^s \underline{u} \leq -f(x, \underline{u}) & \text{dans } \Omega \\ \underline{u} \leq g & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ E\underline{u} = 0 & \text{dans } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta)^s \overline{u} \geq -f(x, \overline{u}) & \text{dans } \Omega \\ \overline{u} \geq g & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ E\overline{u} = 0 & \text{sur } \partial\Omega \end{cases}$$

c'est-à-dire que pour toute $\phi \in \mathcal{T}(\Omega)$, $(-\Delta)^s \phi|_{\Omega} \geq 0$

$$\int_{\Omega} \underline{u} (-\Delta)^s \phi \leq - \int_{\Omega} f(x, \underline{u}) \phi - \int_{\mathbb{R}^N \setminus \overline{\Omega}} g (-\Delta)^s \phi, \quad \int_{\Omega} \overline{u} (-\Delta)^s \phi \geq - \int_{\Omega} f(x, \overline{u}) \phi - \int_{\mathbb{R}^N \setminus \overline{\Omega}} g (-\Delta)^s \phi.$$

On suppose aussi que $\underline{u} \leq \overline{u}$ dans Ω , et $\underline{u}, \overline{u} \in L^\infty(\Omega) \cap C(\Omega)$. Alors le problème nonlinéaire admet une solution faible u qui satisfait

$$\underline{u} \leq u \leq \overline{u}.$$

En plus,

- si f est croissante dans la deuxième variable, i.e. $f(x, s) \leq f(x, t)$ lorsque $s \leq t$, pour toute $x \in \Omega$, alors il existe une seule solution,
- sinon, il y a une seule solution minimale u_1 , ce qui veut dire une solution u_1 telle que $\underline{u} \leq u_1 \leq v$ pour toute sursolution $v \geq \underline{u}$.

Au cas où la donnée au bord g est une fonction positive bornée, alors le Théorème (avec f.2) aussi comme hypothèse) fournit une seule solution parce qu'on peut considérer $\overline{u} = \sup g$ et $\underline{u} = 0$. Ensuite on peut attaquer le problème avec données non bornées, et on est spécialement intéressé aux données avec une explosion sur $\partial\Omega$. L'existence de solution s -harmoniques grandes est la clé pour résoudre tous les théorèmes suivants.

THÉORÈME: CONSTRUCTION DE GRANDES SOLUTIONS | Soit $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ une fonction qui satisfait f.1) and f.2). Ils existent $u, v : \mathbb{R}^N \rightarrow [0, +\infty]$ telles que

$$\left\{ \begin{array}{ll} (-\Delta)^s u = -f(x, u) & \text{dans } \Omega \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} u(x) = +\infty, \end{array} \right. \quad \text{et} \quad \left\{ \begin{array}{ll} (-\Delta)^s v = f(x, v) & \text{dans } \Omega \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} v(x) = +\infty. \end{array} \right.$$

Selon la nature de la nonlinéarité f on peut être plus précis pour ce qui concerne les valeurs de Dirichlet de u .

THÉORÈME | Soit $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ une fonction qui satisfait f.1) et f.2), et $g : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow [0, +\infty]$ une fonction mesurable admissible ; soit aussi $h \in C(\partial\Omega)$, $h \geq 0$. Le problème semilinéaire

$$\left\{ \begin{array}{ll} (-\Delta)^s u(x) = -f(x, u(x)) & \text{dans } \Omega \\ u = g & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = h & \text{sur } \partial\Omega \end{array} \right.$$

satisfait les énoncés suivants:

i si $h \equiv 0$, l'équation a une solution faible pour tout g ,

ii si $h \not\equiv 0$ alors

- le problème a une solution faible si ils existent $a_1, a_2 \geq 0$ et $p \in [0, \frac{1+s}{1-s})$ tels que

$$f(x, t) \leq a_1 + a_2 t^p, \quad \text{pour } t > 0,$$

- le problème n'admet pas de solution si ils existent $b_1, T > 0$ tels que

$$f(x, t) \geq b_1 t^{\frac{1+s}{1-s}}, \quad \text{pour } t > T.$$

Si, en plus, f est croissante dans la deuxième variable alors le problème admet une seule solution.

REMARQUE | À notre connaissance, les seuls résultats disponibles traitant les problèmes semilinéaires pour le Laplacien fractionnaire sont [43] et [25], comme annoncé dans l'introduction. Le théorème au dessus les améliore des façons suivantes :

1. on a des hypothèses assez générales sur la nonlinéarité, alors que dans [43] et [25] considèrent seulement des nonlinéarités du type puissance,
2. on donne la valeur exacte pour la croissance critique (i.e. $f(t) \sim t^{(1+s)/(1-s)}$) qui sépare l'existence et la non existence des solutions,
3. on spécifie en quel sens les solutions explosent au bord via l'opérateur de trace E , de cette manière on explique pourquoi les auteurs dans [43] et [25] trouvent une infinité des solutions.

Par ailleurs, on ne trouve pas *toutes* les solutions avec explosion au bord qui sont traitées dans [43] et [25] : elles correspondent aux données au bord $h \equiv +\infty$ ou bien $g \in L^1(C\Omega)$ non admissible. Cette question sera étudié dans la suite.

THÉORÈME | Soit $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ une fonction qui satisfait f.1) et f.2), et $g : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow [0, +\infty]$ une fonction mesurable admissible; soit aussi $h \in C(\partial\Omega)$, $h \geq 0$. On suppose aussi que

$$f(x, t) \leq \Lambda(t), \quad \text{pour tout } x \in \overline{\Omega}, t \geq 0$$

où $\Lambda \in C^1(0, +\infty)$ est concave et $\Lambda'(t) \xrightarrow{t \uparrow \infty} 0$. Alors il existe une solution faible u au problème semilinéaire

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x)) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = h & \text{on } \partial\Omega. \end{cases}$$

THÉORÈME | Soit $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ une fonction qui satisfait f.1) and f.2). Pour $0 < \beta < 1 - s$, on considère les problèmes

$$\begin{aligned} (\star) \quad & \begin{cases} (-\Delta)^s u(x) = \lambda f(x, u(x)) & \text{dans } \Omega \\ u(x) = \delta(x)^{-\beta} & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = 0 & \text{sur } \partial\Omega \end{cases} \\ (\star\star) \quad & \begin{cases} (-\Delta)^s u(x) = \lambda f(x, u(x)) & \text{dans } \Omega \\ u(x) = 0 & \text{dans } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = 1 & \text{sur } \partial\Omega. \end{cases} \end{aligned}$$

Existence. Si ils existent $a_1, a_2, T > 0$ et $p \geq 1$, tels que

$$f(x, t) \leq a_1 + a_2 t^p, \quad x \in \Omega, t > T.$$

et $p\beta < 1 + s$, alors il existe $L_1 > 0$ qui dépend de β et p tel que le problème (\star) admet une solution faible $u \in L^1(\Omega)$ pour tout $\lambda \in [0, L_1]$. De la même façon, si $p(1 - s) < 1 + s$, il existe $L_2 > 0$ qui dépend de p tel que le problème $(\star\star)$ admet une solution faible $u \in L^1(\Omega)$ pour tout $\lambda \in [0, L_2]$.

Nonexistence. Si ils existent $b, T > 0$ et $q > 0$, tels que

$$bt^q \leq f(x, t), \quad x \in \Omega, t > T.$$

et $q\beta \geq 1 + s$, alors le problème (\star) admet une solution faible seulement pour $\lambda = 0$. De la même façon, si $q(1 - s) \geq 1 + s$, alors le problème $(\star\star)$ admet une solution faible seulement pour $\lambda = 0$.

On remarque finalement, que, avec la définition de solution faible que l'on utilise, la nonexistence implique l'explosion complète :

DÉFINITION | Si pour chaque suite croissante $\{f_k\}_{k \in \mathbb{N}}$ de fonctions continues telles que $f_k \uparrow f$ ponctuellement lorsque $k \uparrow +\infty$, et toute suite $\{u_k\}_{k \in \mathbb{N}}$ de solutions positives

$$\begin{cases} (-\Delta)^s u_k = f_k(x, u_k) & \text{and } \Omega \\ u_k = g & \text{and } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu_k = h & \text{sur } \partial\Omega, \end{cases}$$

on a

$$\lim_{k \uparrow +\infty} \frac{u_k(x)}{\delta(x)^s} = +\infty, \quad \text{uniformement pour } x \in \Omega,$$

alors on dit qu'on a explosion complète.

THÉORÈME | Soit $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ une fonction qui satisfait f.1) and f.2) et $g : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow [0, +\infty]$ une fonction mesurable admissible ; soit aussi $h \in C(\partial\Omega)$, $h \geq 0$. S'il n'y a aucune solution de

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = h & \text{on } \partial\Omega, \end{cases}$$

alors on a explosion complète.

L'intérêt dans des solutions encore plus grandes

Jusque là on a démontré l'existence de solutions avec explosion au bord avec une donnée négative dans l'équation ; de toute façon ce comportement singulier est du à un phénomène linéaire plutôt que à une compensation entre la nonlinéarité et l'explosion (comme dans le cas classique). En effet aucune hypothèse sur f apparaît, sauf lorsque $h \not\equiv 0$ où on a besoin que

$$\int_{\Omega} f(x, \delta(x)^{s-1}) \delta(x)^s dx < \infty$$

pour pouvoir donner un sens à la définition de solution faible L^1 .

Pour cette raison, dans ce qui suit on va étudier l'existence des solutions au problèmes de la forme

$$\begin{cases} (-\Delta)^s u = -f(u) & \text{dans } \Omega \\ u = g & \text{dans } \mathbb{R}^N \setminus \Omega \\ \delta^{1-s} u = +\infty & \text{sur } \partial\Omega \end{cases} \quad g \geq 0, \quad \int_{\mathbb{R}^N \setminus \Omega} g \delta^{-s} \min\{1, \delta^{-N-s}\} = +\infty.$$

et on donnera des conditions suffisantes pour leur solvabilité. En faisant cela, on commence à répondre à la question laissée ouverte dessus. Les résultats dans le Théorèmes 1.21 et 1.23 peuvent être appliqués aussi au problème fractionnaire de Yamabe singulier, voir par exemple González, Mazzeo et Sire [49].

Vers une condition de Keller-Osserman fractionnaire

Dans le Chapitre 4 on va travailler dans les hypothèses suivantes :

- Ω est un domaine ouvert borné de régularité C^2 ,
- f est une fonction C^1 croissante avec $f(0) = 0$,
- F est la primitive de f qui s'annule en 0 :

$$F(t) := \int_0^t f(\tau) d\tau,$$

- ils existent $0 < m < M$, tels que

$$1 + m \leq \frac{tf'(t)}{f(t)} \leq 1 + M,$$

et donc f satisfait la condition de Keller-Osserman parce que en intégrant l'inégalité de gauche, on obtient

$$f(t) \geq f(1)t^{1+m} \quad \text{and} \quad F(t) \geq \frac{f(1)}{2+m} t^{2+m};$$

et on peut aussi définir

$$\phi(u) := \int_u^{+\infty} \frac{dt}{\sqrt{F(t)}},$$

- f satisfait

$$\int_1^{+\infty} \phi(t)^{1/s} dt < +\infty.$$

Dans ce qui suit, on va utiliser l'expression $g \asymp h$ où $g, h : (0, +\infty) \rightarrow (0, +\infty)$ pour raccourcir

$$\text{il existe } C > 0 \text{ tel que } \frac{h(t)}{C} \leq g(t) \leq Ch(t), \text{ pour tout } t > 0.$$

REMARQUE | La fonction $\phi : (0, +\infty) \rightarrow (0, +\infty)$ est décroissante et

$$\lim_{t \downarrow 0} \phi(t) = +\infty, \quad \lim_{t \uparrow +\infty} \phi(t) = 0.$$

En plus

$$\phi'(u) = -\frac{1}{\sqrt{F(u)}}.$$

est du même ordre de $-(uf(u))^{-1/2}$, vu que pour tout $t > 0$ et quelque $\tau \in (0, t)$

$$\frac{F(t)}{tf(t)} = \frac{f(\tau)}{f(\tau) + \tau f'(\tau)} \begin{cases} \geq \frac{1}{2+M} \\ \leq \frac{1}{2+m} \end{cases} \quad \text{par le Théorème de Cauchy.}$$

Cela implique que l'ordre de $\phi(u)$ est le même de $(u/f(u))^{1/2}$, en effet pour $u > 0$ et quelque $t \in (0, u)$

$$\frac{\sqrt{\frac{u}{f(u)}}}{\phi(u)} = \frac{\frac{1}{2}\sqrt{\frac{f(t)}{t}} \cdot \frac{f(t)-tf'(t)}{f(t)^2}}{\phi'(t)} \asymp \frac{f(t)-tf'(t)}{-f(t)} = \frac{tf'(t)}{f(t)} - 1$$

qui appartient à (m, M) d'après l'hypothèse technique sur le rapport entre f et sa dérivée. À noter que l'hypothèse sur l'intégrabilité de f sera équivalente à

$$\int_1^{+\infty} \left(\frac{t}{f(t)} \right)^{\frac{1}{2s}} dt < +\infty.$$

REMARQUE | Dans [56] et [70] il a été montré que la condition de Keller-Osserman est nécessaire et suffisante pour l'existence d'une solution de

$$\begin{cases} -\Delta u = -f(u) \text{ dans } \Omega, \\ \lim_{x \rightarrow \partial\Omega} u(x) = +\infty. \end{cases}$$

Si l'on pose $s = 1$ dans l'hypothèse sur l'intégrabilité de f

$$+\infty > \int_u^{+\infty} \phi(t) dt \asymp \int_u^{+\infty} \sqrt{\frac{t}{f(t)}} dt \asymp \int_u^{+\infty} \frac{t}{\sqrt{F(t)}} dt$$

on obtient la condition pour forcer la solution classique u à être dans $L^1(\Omega)$. En effet, dans [40, Théorème 1.6] il est prouvé que une solution u satisfait

$$\lim_{x \rightarrow \partial\Omega} \frac{\phi(u(x))}{\delta(x)} = 1$$

qui implique que $u \in L^1(\Omega)$ si et seulement si ϕ^{-1} , la fonction inverse de ϕ (on rappelle qu'elle est décroissante), est intégrable dans un voisinage de 0, c'est-à-dire que avec un changement de variable

$$+\infty > \int_0^\eta \phi^{-1}(r) dr = \int_{\phi^{-1}(\eta)}^{+\infty} t|\phi'(t)| dt = \int_{t_0}^{+\infty} \frac{t}{\sqrt{F(t)}} dt.$$

Nos résultats principaux peuvent être récapitulés ainsi.

THÉORÈME | On suppose que le terme nonlinéaire f satisfait les hypothèses au dessus et

$$\int_{t_0}^{+\infty} f(t)t^{-2/(1-s)} dt < +\infty.$$

Alors le problème

$$\begin{cases} (-\Delta)^s u = -f(u) & \text{dans } \Omega \\ u = 0 & \text{dans } \mathbb{R}^N \setminus \Omega \\ \delta^{1-s} u = +\infty & \text{sur } \partial\Omega \end{cases}$$

admet une solution $u \in L^1(\Omega)$. En plus il existe $c > 0$ avec lequel

$$\phi(u(x)) \geq c\delta(x)^s \quad \text{près de } \partial\Omega.$$

REMARQUE | La condition $u \in L^1(\Omega)$ est nécessaire pour donner un sens au Laplacien fractionnaire, comme toutes les différentes définitions indiquent (voir au dessus). À remarquer aussi le comportement asymptotique donné par le Théorème et celui de la solution classique.

THÉORÈME | On suppose que le terme nonlinéaire f satisfait les hypothèses au dessus et

$$\begin{aligned} g : \mathbb{R}^N \setminus \Omega &\longrightarrow [0, +\infty), & g &\in L^1(\mathbb{R}^N \setminus \Omega) \\ \phi(g(x)) &\geq \delta(x)^s, & &\text{près de } \partial\Omega. \end{aligned}$$

Le problème

$$\begin{cases} (-\Delta)^s u = -f(u) & \text{dans } \Omega \\ u = g & \text{dans } \mathbb{R}^N \setminus \Omega \end{cases}$$

admet une solution $u \in L^1(\Omega)$. En plus, il existe $c > 0$ avec lequel

$$\phi(u(x)) \geq c\delta(x)^s \quad \text{près de } \partial\Omega.$$

À remarquer que dans ce dernier problème on ne prescrit pas la trace singulière sur $\partial\Omega$.

PARTIE II : GRANDES SOLUTIONS POUR $(-\Delta|_\Omega)^s$

Étant donné un domaine borné Ω de l'espace \mathbb{R}^N , l'opérateur *Laplacien fractionnaire spectral* $(-\Delta|_\Omega)^s$ pour $s \in (0, 1)$ classiquement est défini comme une puissance fractionnaire du Laplacien couplé avec des conditions de Dirichlet homogènes, vu comme un opérateur auto-adjoint sur l'espace de Lebesgue $L^2(\Omega)$. Cela donne un opérateur nonlocal de type elliptique avec conditions de Dirichlet homogènes. Bibliographie récente sur cet opérateur peut être trouvée dans Bonforte, Sire et Vazquez [14], Grubb [50], Caffarelli et Stinga [20], Servadei et Valdinoci [77].

Un aspect de la théorie n'est toujours pas exploré : la formulation de conditions nonhomogènes au bord naturelles. On trouve un premier essai dans le travail de Dhifli, Mâagli et Zribi [36]. Les recherches qu'on a développées dans cette Partie donnent quelques réponses dans cette direction. On va donner une formulation faible, qui est bien posée en sens de Hadamard, pour les problèmes linéaires de la forme

$$\begin{cases} (-\Delta|_\Omega)^s u = \mu & \text{in } \Omega, \\ \frac{u}{h_1} = \zeta & \text{on } \partial\Omega, \end{cases}$$

où h_1 est une fonction de référence avec un comportement singulier au bord. Plus précisément, h_1 est bornée entre deux multiples de $\delta^{-(2-2s)}$ et la condition au bord doit être interprétée comme une limite

lorsque δ converge à zero. Autrement dit, contrairement à ce que l'on a dans le problème de Dirichlet pour l'opérateur de Laplace, les conditions au bord doivent être singulières. En plus, si les données μ, ζ sont lisses, la solution explose⁹ à la vitesse fixée $\delta^{-(2-2s)}$. En effet, dans le cas particulier des fonctions s -harmoniques positives, lorsque $\mu = 0$, la condition au bord singulière était déjà identifiée dans des travaux précédents qui insistaient sur les aspects probabilistes et de théorie du potentiel : voir par exemple Song et Vondraček [82], Glover, Pop-Stojanovic, Rao, Šikić, Song and Vondraček [47] et Song [81].

Dès que l'on s'occupe des problèmes nonlinéaires, des conditions encore plus singulières peuvent être atteintes : si $\mu = -u^p$, pour certaines valeurs de p on pourrait choisir $\zeta = +\infty$, et la solution aura une explosion au bord d'ordre supérieur par rapport à $\delta^{-(2-2s)}$ et contrôlé par le comportement $\delta^{-2s/(p-1)}$. À noter que la valeur $\zeta = +\infty$ n'est pas admissible dans un problème linéaire. Cela a été observé dans le contexte du Laplacien fractionnaire et c'est ce que l'on montrera aussi pour le Laplacien fractionnaire spectral. Il est intéressant de remarquer que les exposants admissibles p sont différents selon l'opérateur que l'on considère.

Résultats principaux

On rappelle d'abord les définitions qu'on va utiliser.

DÉFINITION | Soit $\Omega \subset \mathbb{R}^N$ un domaine borné et soit $\{\varphi_j\}_{j \in \mathbb{N}}$ une base hilbertienne de $L^2(\Omega)$ constituée par les fonctions propres du Laplacien de Dirichlet $-\Delta|_{\Omega}$, associées aux valeurs propres λ_j , $j \in \mathbb{N}$, i.e. $\varphi_j \in H_0^1(\Omega) \cap C^\infty(\Omega)$ et $-\Delta\varphi_j = \lambda_j\varphi_j$ dans Ω . Étant donné $s \in (0, 1)$, on considère l'espace de Hilbert

$$H(2s) := \left\{ v = \sum_{j=1}^{\infty} \widehat{v}_j \varphi_j \in L^2(\Omega) : \|v\|_{H(2s)}^2 = \sum_{j=0}^{\infty} \lambda_j^{2s} |\widehat{v}_j|^2 < \infty \right\}.$$

Le Laplacien fractionnaire spectral de $u \in H(2s)$ est la fonction de $L^2(\Omega)$ donnée par la formule

$$(-\Delta|_{\Omega})^s u = \sum_{j=1}^{\infty} \lambda_j^s \widehat{u}_j \varphi_j.$$

À noter que $C_c^\infty(\Omega) \subset H(2s) \hookrightarrow L^2(\Omega)$. Donc l'opérateur $(-\Delta|_{\Omega})^s$ est non borné, défini sur un sous-espace dense et avec inverse borné $(-\Delta|_{\Omega})^{-s}$ dans $L^2(\Omega)$. Autrement, pour tout $x \in \Omega$,

$$(-\Delta|_{\Omega})^s u(x) = PV \int_{\Omega} [u(x) - u(y)] J(x, y) dy + \kappa(x) u(x),$$

où, si $p_{\Omega}(t, x, y)$ denote le noyau de la chaleur de $-\Delta|_{\Omega}$,

$$J(x, y) = \frac{s}{\Gamma(1-s)} \int_0^\infty \frac{p_{\Omega}(t, x, y)}{t^{1+s}} dt \quad \text{and} \quad \kappa(x) = \frac{s}{\Gamma(1-s)} \int_{\Omega} \left(1 - \int_{\Omega} p_{\Omega}(t, x, y) dy \right) \frac{dt}{t^{1+s}}$$

sont respectivement¹⁰ le noyau de saut et la mesure d'arrêt, voir Song et Vondraček [82, formules (3.3) et (3.4)]. On suppose pour le reste du Chapitre que

$$\Omega \text{ est de classe } C^{1,1}.$$

En particulier, des bornes précises pour p_{Ω} sont connues et donnent des estimations précises pour $J(x, y)$, d'une façon telle que les intégrales restent bien définies pour $u \in C_{loc}^{2s+\varepsilon}(\Omega) \cap L^1(\Omega, \delta(x)dx)$ pour quelque $\varepsilon > 0$. Cela permet de définir le Laplacien fractionnaire spectral sur des fonctions qui ne s'annulent pas sur le bord de Ω . Par exemple $u(x) \equiv 1$, $x \in \Omega$, n'appartient pas à $H(2s)$ si $s \geq 1/4$, mais elle résout le problème au dessus pour $\mu = \kappa$ et $\zeta = 0$ quoi qu'il en soit.

⁹Ceci est très proche à la théorie de conditions nonhomogènes pour le Laplacien fractionnaire - même si dans ce cas là l'explosion est d'ordre $\delta^{-(1-s)}$ - comme analysé dans la Partie I de ce manuscrit et d'une perspective différente par Grubb [50].

¹⁰avec le langage de la théorie du potentiel d'un processus stochastique arrêté. La première intégrale doit être interprétée au sens de la valeur principale.

DÉFINITION | La fonction de Green et le noyau de Poisson du Laplacien fractionnaire spectral sont définis par

$$G_{|\Omega}^s(x, y) = \frac{1}{\Gamma(s)} \int_0^\infty p_\Omega(t, x, y) t^{s-1} dt, \quad x, y \in \Omega, x \neq y, s \in (0, 1],$$

et par

$$P_{|\Omega}^s(x, y) := - \frac{\partial}{\partial \nu_y} G_{|\Omega}^s(x, y), \quad x \in \Omega, y \in \partial\Omega.$$

où ν est la dérivée normale extérieure sur $\partial\Omega$.

Dans la Section 7.1, on va montrer que $P_{|\Omega}^s$ est bien défini (voir Lemme 7.4) et on va réviser des identités sur la fonction de Green $G_{|\Omega}^s$ et le noyau de Poisson $P_{|\Omega}^s$.

DÉFINITION | On considère l'espace de fonctions test

$$\mathcal{T}(\Omega) := (-\Delta|_\Omega)^{-s} C_c^\infty(\Omega)$$

et le poids

$$h_1(x) = \int_{\partial\Omega} P_{|\Omega}^s(x, y) d\sigma(y), \quad x \in \Omega.$$

Étant données deux mesures de Radon $\mu \in \mathcal{M}(\Omega)$ et $\zeta \in \mathcal{M}(\partial\Omega)$ avec

$$\int_\Omega \delta(x) d|\mu|(x) < \infty, \quad |\zeta|(\partial\Omega) < \infty,$$

une fonction $u \in L_{loc}^1(\Omega)$ est une solution faible de

$$\begin{cases} (-\Delta|_\Omega)^s u = \mu & \text{dans } \Omega \\ \frac{u}{h_1} = \zeta & \text{sur } \partial\Omega \end{cases}$$

si pour tout $\psi \in \mathcal{T}(\Omega)$

$$\int_\Omega u (-\Delta|_\Omega)^s \psi = \int_\Omega \psi d\mu - \int_{\partial\Omega} \frac{\partial \psi}{\partial \nu} d\zeta.$$

On montrera que $\mathcal{T}(\Omega) \subseteq C_0^1(\overline{\Omega})$, voir Lemme 7.18, et donc les intégrales au dessus sont bien définies. La définition qu'on vient d'énoncer est une formulation faible du problème comme le Lemme suivant dit.

LEMMA | 1. (solutions faible sont solutions au sens des distributions) On suppose que $u \in L_{loc}^1(\Omega)$ est une solution faible. Alors $u \in L^1(\Omega, \delta(x)dx)$ et $(-\Delta|_\Omega)^s u = \mu$ au sens des distributions i.e. pour tout $\psi \in C_c^\infty(\Omega)$, $\frac{(-\Delta|_\Omega)^s \psi}{\delta}$ est borné et

$$\int_\Omega u (-\Delta|_\Omega)^s \psi = \int_\Omega \psi d\mu,$$

2. (si les données sont lisses, la solution faible est classique) On suppose que $u \in L_{loc}^1(\Omega)$ est une solution faible, où $\mu \in C^\alpha(\overline{\Omega})$ pour quelque α tel que $\alpha + 2s \notin \mathbb{N}$ et $\zeta \in C(\partial\Omega)$. Alors $(-\Delta|_\Omega)^s u$ est bien défini ponctuellement pour tout $x \in \Omega$, $(-\Delta|_\Omega)^s u(x) = \mu(x)$ pour tout $x \in \Omega$ et pour tout $x_0 \in \partial\Omega$,

$$\lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} \frac{u(x)}{h_1(x)} = \zeta(x_0).$$

3. (les solutions classique sont faibles) On suppose que $u \in C_{loc}^{2s+\varepsilon}(\Omega)$ est telle que $u/h_1 \in C(\overline{\Omega})$. Soit $\mu = (-\Delta|_\Omega)^s u$ et $\zeta = u/h_1|_{\partial\Omega}$. Alors u est une solution faible.

On va présenter les fonctions harmoniques dans la Section 7.2 en regardant leur trace au bord dans la Section 7.3. En effet on peut montrer

THÉORÈME | Soit $\zeta \in L^1(\partial\Omega)$ et

$$u(x) = \int_{\partial\Omega} P_{|\Omega}^s(x, y) d\zeta(y).$$

Alors pour tout $\phi \in C^0(\overline{\Omega})$ on a

$$\frac{1}{t} \int_{\{\delta(x) \leq t\}} \frac{\phi(x)}{h_1(x)} \left(\int_{\partial\Omega} P_{|\Omega}^s(x, y) \zeta(y) d\sigma(y) \right) dx \xrightarrow[t \downarrow 0]{} \int_{\partial\Omega} \phi(y) \zeta(y) d\sigma(y).$$

En plus on a aussi

THÉORÈME | Pour tout $\mu \in \mathcal{M}(\Omega)$, telle que

$$\int_{\Omega} \delta d|\mu| < \infty,$$

et tout $\phi \in C^0(\overline{\Omega})$ on a

$$\frac{1}{t} \int_{\{\delta(x) \leq t\}} \frac{\phi(x)}{h_1(x)} \left(\int_{\Omega} G_{|\Omega}^s(x, y) d\mu(y) \right) dx \xrightarrow[t \downarrow 0]{} 0.$$

On va montrer que le problème linéaire est bien-posé dans la Section 7.4, et plus précisément

THÉORÈME | Étant données deux mesures de Radon $\mu \in \mathcal{M}(\Omega)$ et $\zeta \in \mathcal{M}(\partial\Omega)$ il existe une seule fonction $u \in L^1_{loc}(\Omega)$ qui soit solution faible du problème au dessus. En plus, pour presque tout $x \in \Omega$,

$$u(x) = \int_{\Omega} G_{|\Omega}^s(x, y) d\mu(y) + \int_{\partial\Omega} P_{|\Omega}^s(x, y) d\zeta(y)$$

et

$$\|u\|_{L^1(\Omega, \delta(x)dx)} \leq C(\Omega, N, s) (\|\delta\mu\|_{\mathcal{M}(\Omega)} + \|\zeta\|_{\mathcal{M}(\partial\Omega)}).$$

On a aussi les estimations suivantes :

$$\begin{aligned} \|u\|_{L^p(\Omega, \delta(x)dx)} &\leq C_1 \|\delta\mu\|_{\mathcal{M}(\Omega)} & \text{si } \zeta = 0 \text{ et } p \in \left[1, \frac{N+1}{N+1-2s}\right) \\ \|u\|_{C^\alpha(\overline{\omega})} &\leq C_2 (\|\mu\|_{L^\infty(\omega)} + \|\zeta\|_{\mathcal{M}(\partial\Omega)}) & \text{si } \omega \subset\subset \Omega \text{ et } \alpha \in (0, 2s) \\ \|u\|_{C^{2s+\alpha}(\overline{\omega})} &\leq C_3 (\|\mu\|_{C^\alpha(\overline{\omega})} + \|\zeta\|_{\mathcal{M}(\partial\Omega)}) & \text{si } \omega \subset\subset \Omega \text{ et } 2s + \alpha \notin \mathbb{N}. \end{aligned}$$

où $C_1 = C_1(\Omega, N, s, p)$, $C_2 = C_2(\Omega, \omega, N, s, \alpha)$, $C_3 = C_3(\Omega, \omega, N, s, \alpha)$.

Dans la Section 8.1 on résout les problèmes nonlinéaires.

THÉORÈME | Soit $g(x, t) : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ une fonction de Carathéodory et $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ une fonction croissante tels que $g(x, 0) = 0$ et pour presque tout $x \in \Omega$ et tout $t > 0$,

$$0 \leq g(x, t) \leq h(t) \quad \text{où} \quad h(\delta^{-(2-2s)})\delta \in L^1(\Omega).$$

Alors le problème

$$\begin{cases} (-\Delta|_{\Omega})^s u = -g(x, u) & \text{dans } \Omega \\ \frac{u}{h_1} = \zeta & \text{sur } \partial\Omega \end{cases}$$

a une solution $u \in L^1(\Omega, \delta(x)dx)$ pour tout $\zeta \in C(\partial\Omega)$, $\zeta \geq 0$. Si $t \mapsto g(x, t)$ est croissante alors la solution est unique.

Enfin, dans la Section 8.2, on va prouver que

THÉORÈME | Soit

$$p \in \left(1 + s, \frac{1}{1-s}\right).$$

Alors il existe une fonction $u \in L^1(\Omega, \delta(x)dx) \cap C^\infty(\Omega)$ qui résout

$$\begin{cases} (-\Delta|_\Omega)^s u = -u^p & \text{dans } \Omega, \\ \frac{u}{h_1} = +\infty & \text{sur } \partial\Omega \end{cases}$$

au sens suivant : la première égalité est ponctuelle et au sense des distributions, la condition au bord est une limite ponctuelle. Il existe aussi une constante $C = C(\Omega, N, s, p)$ telle que

$$0 \leq u \leq C\delta^{-\frac{2s}{p-1}}.$$

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Ai miei genitori, e soprattutto al papà, voglio dire di non aver paura: chiaro il desiderio, chiara la strada... *quando eri più giovane ti cingevi la veste da solo, e andavi dove volevi; ma quando sarai vecchio tenderai le tue mani, e un altro ti cingerà la veste e ti porterà dove tu non vuoi*. Grazie a mio fratello e alle mie sorelle, perché “hai fatto il nostro cuore irrequieto finché non riposa in te”.

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To my wife,
perché nulla può abbattere la casa
fondata sopra la roccia. (Mt 7,25)

Lady, by one light only
We look from Alfred's eyes,
We know he saw athwart the wreck
The sign that hangs about your neck,
Where One more than Melchizedek
Is dead and never dies.

Therefore I bring these rhymes to you
Who brought the cross to me,
Since on you flaming without flaw
I saw the sign that Guthrum saw
When he let break his ships of awe,
And laid peace on the sea. [...]

And I thought "I will go with you,
As man with God has gone,
And wander with a wandering star,
The wandering heart of things that are,
The fiery cross of love and war
That like yourself, goes on." [...]

Take this; in memory of the hour
we strayed a space from home
and saw the smoke-hued hamlets, quaint
with Westland king and Westland saint
and watched the western glory faint
along the road to Frome.

(G.K. Chesterton, *The Ballad of the White Horse*,
Dedication, vv. 47–58; 65–70; 89–94)

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Introduction

This manuscript presents the study of boundary blow-up solutions for two different integro-differential operators, both called *fractional Laplacian*. Operators of this sort have been intensively studied in recent years, since they show many resemblances with the classical theory of elliptic operators, whereas they are often defined as integral operators with singular kernels and long-range interactions, making them *nonlocal* operators¹¹. At the same time, they naturally appear in concrete applications, most of all to model phenomena where long-range interactions are present. We have for example: phase transition models [79], crystal dislocation [84], the obstacle problem [78], flame propagation [80], minimal surfaces [17], finance [29], and materials science [9].

The fractional Laplacian. We call the first one of these operators simply¹² *fractional Laplacian* and we denote it by $(-\Delta)^s$. Fixed $s \in (0, 1)$, for a function¹³ $u : \mathbb{R}^N \rightarrow \mathbb{R}$ this operator can be defined in a number of different equivalent ways:

- as an integral operator with singular kernel [37]

$$(-\Delta)^s u(x) = C_{N,s} \text{ p.v. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy \quad (2)$$

where “p.v.” stands for the integral in the principal value sense, $C_{N,s}$ is a positive renormalizing constant and it equals (e.g. [74, formula (A.1)] and [34, formula (1.3) and following])

$$C_{N,s} = \frac{4^s \Gamma(\frac{N}{2} + s)}{\pi^{N/2}} \cdot \frac{s}{\Gamma(1 - s)}; \quad (3)$$

- as a pseudodifferential operator [37, 50]

$$\mathcal{F} [(-\Delta)^s u](\xi) = |\xi|^{2s} \mathcal{F} u(\xi) \quad (4)$$

where \mathcal{F} denotes the Fourier transform;

- as a Dirichlet-to-Neumann map [18]: let $U : \mathbb{R}^N \times [0, +\infty)$ solve

$$\begin{cases} \operatorname{div} (t^{1-2s} U(x, t)) = 0 & x \in \mathbb{R}^N, t > 0 \\ U(x, 0) = u(x) & x \in \mathbb{R}^N \end{cases}$$

then

$$(-\Delta)^s u(x) = -c(N, s) \lim_{t \downarrow 0} t^{1-2s} \partial_t U(x, t) \quad (5)$$

where $c(N, s)$ is the positive renormalizing constant

$$c(N, s) = \frac{4}{(N + 2 - 2s) \pi^{N/2+s} \Gamma(N/2 + s)}; \quad (6)$$

¹¹Note that all differential operators, such as the classical Laplacian $-\Delta$, are purely local.

¹²It is sometimes called *restricted fractional Laplacian* when restricted only to functions who are supported in a fixed bounded region, see for example [86].

¹³It is not our purpose in this Introduction to provide thorough regularity or integrability assumptions on u in order to make sense of the definitions: let us work only formally at this stage.

- as the infinitesimal generator of the heat semigroup $e^{t\Delta}$ in \mathbb{R}^N subordinated in the sense of Bochner with corresponding Bernstein function $\tau \mapsto \tau^s$ [76, Section 13.1]

$$(-\Delta)^s u(x) = \frac{s}{\Gamma(1-s)} \int_0^\infty \left(\frac{u(x) - e^{t\Delta} u(x)}{t} \right) \frac{dt}{t^s}; \quad (7)$$

- as the infinitesimal generator of a symmetric α -stable stochastic process $(X_t)_{t \geq 0}$ for $\alpha = 2s$

$$(-\Delta)^s u(x) = \lim_{t \downarrow 0} \frac{u(x) - \mathbb{E}_x[u(X_t)]}{t}. \quad (8)$$

The operator turns out to be “nonlocal” in the sense that to compute it (on a given function and at a fixed point) it is not sufficient to know the values of the function in a neighbourhood of the point; conversely, modifying the function far away from the point affects the value at the point itself (this feature is clear when using the representation in (2)); this is not at all the case, for instance, for the classical Laplacian $-\Delta$. The different, yet equivalent, definitions for $(-\Delta)^s$ allow to take very different approaches in solving related problems, creating a fruitful interplay between e.g. variational techniques, the theory of pseudodifferential operators, functional analysis and potential theory.

The spectral fractional Laplacian. The second operator we deal with is another nonlocal operator whose definition strongly depends on the domain taken into account. So, fix a bounded domain $\Omega \subset \mathbb{R}^N$, $s \in (0, 1)$ and denote by $\delta(x) = \text{dist}(x, \partial\Omega)$. For a function¹⁴ $u : \Omega \rightarrow \mathbb{R}$ the operator $(-\Delta|_\Omega)^s$ can be defined

- as an integral operator with singular kernel [82]

$$(-\Delta|_\Omega)^s u(x) = p.v. \int_\Omega [u(x) - u(y)] J(x, y) dy + \kappa(x) u(x), \quad x \in \Omega \quad (9)$$

where $J(x, y)$ and $\kappa(x)$ are the functions given by

$$\begin{aligned} J(x, y) &= \frac{s}{\Gamma(1-s)} \int_0^\infty \frac{p_\Omega(t, x, y)}{t^{1+s}} dt \quad \asymp \frac{\delta(x)\delta(y)}{|x-y|^{N+2s}} \cdot \frac{1}{|x-y|^2 + \delta(x)\delta(y)} \\ \kappa(x) &= \frac{s}{\Gamma(1-s)} \int_0^\infty \left(1 - \int_\Omega p_\Omega(t, x, y) dy \right) \frac{dt}{t^{1+s}} \quad \asymp \frac{1}{\delta(x)^{2s}} \end{aligned}$$

and $p_\Omega(t, x, y)$ is the heat kernel of Ω associated to homogeneous boundary conditions;

- via the spectral decomposition of u : given $\{\varphi_j\}_{j \in \mathbb{N}}$ a Hilbert basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Dirichlet Laplacian $-\Delta|_\Omega$ associated to the eigenvalues λ_j , $j \in \mathbb{N}$

$$(-\Delta|_\Omega)^s u(x) = \sum_{j=1}^\infty \lambda_j^s \hat{u}_j \varphi_j(x), \quad \hat{u}_j = \int_\Omega u_j \varphi_j; \quad (10)$$

- as a Dirichlet-to-Neumann map [16, 23]: let $U : \Omega \times [0, +\infty)$ solve

$$\begin{cases} \operatorname{div}(t^{1-2s} U(x, t)) = 0 & x \in \Omega, t > 0 \\ U(x, t) = 0 & x \in \partial\Omega, t > 0 \\ U(x, 0) = u(x) & x \in \Omega \end{cases}$$

then

$$(-\Delta|_\Omega)^s u(x) = -c(N, s) \lim_{t \downarrow 0} t^{1-2s} \partial_t U(x, t) \quad (11)$$

where $c(N, s)$ is the positive renormalizing constant defined in (6);

¹⁴As before, let us work only formally for the moment.

- as the infinitesimal generator of the heat semigroup $e^{t\Delta|_\Omega}$ in Ω subordinated in the sense of Bochner with corresponding Bernstein function $\tau \mapsto \tau^s$ [76, Section 13.1] (and also [47])

$$(-\Delta|_\Omega)^s u(x) = \frac{s}{\Gamma(1-s)} \int_0^\infty \left(\frac{u(x) - e^{t\Delta|_\Omega} u(x)}{t} \right) \frac{dt}{t^s}; \quad (12)$$

- as the infinitesimal generator of a subordinate killed Brownian motion $(\tilde{X}_t)_{t \geq 0}$

$$(-\Delta|_\Omega)^s u(x) = \lim_{t \downarrow 0} \frac{u(x) - \mathbb{E}_x[u(\tilde{X}_t)]}{t}. \quad (13)$$

To our knowledge, definition (10) is the most commonly used in the PDE framework. Still it bears some restrictions on the domain of the operator, and no theory for nonhomogeneous boundary conditions for $(-\Delta|_\Omega)^s$ was available until this work: this is one of the contributions of this thesis, see Chapter 7. This is why we will mostly exploit definitions (9) and (12).

Some stochastic interpretation and the names of the operators. As mentioned above, both operators reveal a deep connection with the theory of stochastic processes. Indeed they generate Lévy processes with jumps¹⁵, i.e. whose paths are discontinuous. The operator $(-\Delta)^s$ generates a so called $(2s)$ -stable process in \mathbb{R}^N , and in particular the distribution law at a fixed time is homogeneous of order $2s$. This process can be built also by subordinating a Brownian motion in time: given a Brownian motion (a Wiener process) $(B_t)_{t \geq 0}$, one evaluates it at random times $(T_t)_{t \geq 0}$ rather than at the deterministic time, which are almost surely increasing according to a linear $(2s)$ -stable process. This way we provide a new process $(B_{T_t})_{t \geq 0}$. The random time T_t has not continuous trajectories in general. In this approach, we finally obtain a Lévy process whose jumps are due to jumps taken in time, as we were recording the random movement of a particle but we stop our camera randomly and then turn it on again (and we might also speed up and slow down the time flow). Recall that the generator of the Brownian motion is the classical (negative) Laplacian $-\Delta$. If the subordinator is well chosen, then the stochastic operation of subordinating has as analytical counterpart taking a fractional power of the generator: at this point $(-\Delta)^s$ comes forth.

The same kind of construction works for $(-\Delta|_\Omega)^s$ but with an interesting difference. Before subordinating, one kills the process upon leaving the domain Ω . For this reason we have written that it generates a *subordinate killed Brownian motion*. The killed process is generated by the Dirichlet Laplacian $-\Delta|_\Omega$ and the subordination provides its fractional power $(-\Delta|_\Omega)^s$. For this reason we have chosen this notation for the spectral fractional Laplacian rather than the common one A_s .

Let us finally mention that we call it *spectral fractional Laplacian* to comply with the usual way it is referred to in the bibliography. We might call it also *fractional Dirichlet Laplacian*, and this would take more into account the way it is constructed. The word *spectral* might be misleading, since it recalls the definition via the spectral decomposition which has some natural restrictions and does not apply to the full class of functions we consider here. In this spirit we might also call *Dirichlet fractional Laplacian* the operator obtained by restricting the fractional Laplacian to function with null boundary conditions. Indeed, such an operator generates a *killed subordinate Brownian motion*, i.e. the process obtained by killing the $(2s)$ -stable Lévy process upon leaving a fixed domain. Mind that “killed subordinate” and “subordinate killed” are different in nature and in the same way “Dirichlet fractional” and “fractional Dirichlet” would not be the same thing.

Boundary blow-up solutions. In the theory of semilinear elliptic equations, functions solving

$$-\Delta u + f(u) = 0, \quad \text{in some } \Omega \subseteq \mathbb{R}^N \text{ open and bounded} \quad (14)$$

¹⁵This type of processes are sometimes called Lévy flights in physics.

coupled with the boundary condition

$$\lim_{x \rightarrow \partial\Omega} u(x) = +\infty$$

are known as *boundary blow-up solutions* or *large solutions*. There is a huge amount of bibliography dealing with this problem which dates back to the seminal work of Bieberbach [10], for $N = 2$ and $f(u) = e^u$. Keller [56] and Osserman [70] independently established a sufficient and necessary condition on a nondecreasing nonlinearity f for the existence of a boundary blow-up solution: it takes the form

$$\int^{+\infty} \frac{dt}{\sqrt{F(t)}} < +\infty, \quad \text{where } F' = f \geq 0 \quad (15)$$

and it is known as *Keller-Osserman condition*. One can find these solutions with singular behaviour at the boundary in a number of applications: for example, Loewner and Nirenberg [63] studied the case $f(u) = u^{(N+2)/(N-2)}$, $N \geq 3$, which is strictly related to the *singular Yamabe problem* in conformal geometry, while Labutin [59] completely characterized the class of sets Ω that admit a large solution for $f(u) = u^q$, $q > 1$, with capacitary methods inspired by the theory of *spatial branching processes*, that are particular stochastic processes; see also the purely probabilistic works by Le Gall [61] and Dhersin and Le Gall [35] dealing with $q = 2$ and the delicate case of nonsmooth domains. Mselati [68] completely classified positive solutions in terms of their boundary trace, which can be $+\infty$ on one part of the boundary and a measure that does not charge sets of zero boundary capacity on the remaining part; see also Dumont, Dupaigne, Goubet and Rădulescu [40] for the case of oscillating nonlinearity. We refer to the very recent book by Marcus and Véron [65, Chapters 5 and 6] for further readings and developments in this direction.

Large s -harmonic functions, integration by parts formulas and weak L^1 theories. In the fractional context, our starting point is that boundary blow-up solutions arise even in linear problems. In particular, it is possible to provide *large s -harmonic functions*, i.e. functions satisfying

$$\left\{ \begin{array}{ll} (-\Delta)^s u = 0 & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} u(x) = +\infty \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} (-\Delta|_{\Omega})^s u = 0 & \text{in } \Omega \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} u(x) = +\infty \end{array} \right. \quad (16)$$

see e.g. Lemmas 2.11, 7.10 and (7.20) below.

These examples show on the one hand how problems for $(-\Delta)^s$ where only outer values are prescribed are ill-posed in the classical sense and different kinds of boundary conditions have to be taken into account, while for $(-\Delta|_{\Omega})^s$ they show the need for nonzero boundary conditions. This means in turn that we need notions of weak solutions that can deal with these new boundary data. We provide answers to these questions respectively in Section 2.2 and Theorem 6.5: the trace is a limit at the boundary weighted with powers of the distance to the boundary, $\delta(x)^{1-s}u(x)$ in the case of the fractional Laplacian and $\delta(x)^{2-2s}u(x)$ in the spectral fractional Laplacian case. We refer also to Grubb [50, e.g. Theorems 4 and 5] for very recent results on the fractional Laplacian, while we mention the work by Dhifli, Mâagli and Zribi [36] for a first attempt on the spectral fractional Laplacian.

In both cases we needed to develop a weak L^1 -theory of Stampacchia's sort with measure data, able to treat this class of s -harmonic functions as well. We consider the following nonhomogeneous problems

$$\left\{ \begin{array}{ll} (-\Delta)^s u = \lambda & \text{in } \Omega \\ u = \mu & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ \frac{u}{w_1} = \nu & \text{on } \partial\Omega \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} (-\Delta|_{\Omega})^s u = \xi & \text{in } \Omega \\ \frac{u}{w_2} = \zeta & \text{on } \partial\Omega \end{array} \right.$$

and we show how are well-posed when $\lambda, \xi \in \mathcal{M}(\Omega)$, $\mu \in \mathcal{M}(\mathbb{R}^N \setminus \overline{\Omega})$, $\nu, \zeta \in \mathcal{M}(\partial\Omega)$ are Radon measures satisfying assumptions

$$\begin{aligned} \int_{\Omega} \delta^s d|\lambda| &< +\infty, & \int_{\Omega} \delta d|\xi| &< +\infty \\ \int_{\mathbb{R}^N \setminus \overline{\Omega}} \delta^{-s} \min\{1, \delta^{-N-s}\} d|\mu| &< +\infty \\ |\nu|(\partial\Omega) &< +\infty, & |\zeta|(\partial\Omega) &< +\infty \end{aligned}$$

and w_1 and w_2 are reference functions¹⁶ defined in Ω whose behaviour is comparable respectively with $\delta^{-(1-s)}$ and $\delta^{-(2-2s)}$. The solutions are functions satisfying respectively

$$\begin{aligned} \int_{\Omega} u(-\Delta)^s \phi &= \int_{\Omega} \phi d\lambda - \int_{\mathbb{R}^N \setminus \Omega} (-\Delta)^s \phi d\mu + \int_{\partial\Omega} \frac{\phi}{\delta^s} \Big|_{\partial\Omega} d\nu & u \in L^1(\Omega), \phi \in \mathcal{T}_1(\Omega), \\ \int_{\Omega} u(-\Delta|_{\Omega})^s \phi &= \int_{\Omega} \phi d\xi - \int_{\partial\Omega} \frac{\partial\phi}{\partial\nu} d\zeta & u \in L^1(\Omega, \delta(x)dx), \phi \in \mathcal{T}_2(\Omega) \end{aligned}$$

and the test function spaces are defined as

$$\begin{aligned} \mathcal{T}_1(\Omega) &= \left\{ \phi \in C^s(\mathbb{R}^N) : \begin{cases} (-\Delta)^s \phi = \psi & \text{in } \Omega, \psi \in C_c^\infty(\Omega) \\ \phi = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ \delta^{1-s} \phi = 0 & \text{on } \partial\Omega \end{cases} \right\}, \\ \mathcal{T}_2(\Omega) &= \{ \phi \in C_0^1(\overline{\Omega}) : (-\Delta|_{\Omega})^s \phi = \psi \text{ in } \Omega, \psi \in C_c^\infty(\Omega) \}. \end{aligned}$$

These results are contained respectively in Sections 2.3 and 7.4. These definitions are inspired by and, in a sense, extend some integration by parts formulas. The techniques are mainly based on potential analysis, exploiting the representation via the Green functions, Poisson and Martin kernels, whose constructions are included in the treatment. A more precise presentation of the results and the rigorous statements of the main findings are included in the introductory Chapters 1 and 6.

Nonlinear problems. So we have seen how boundary singularities can show up even in presence of quite smooth data. When turning to nonlinear problems, this lack of smoothness at the boundary provide a wide class of solutions with a *prescribed* singularity at the boundary, presented in Chapter 3 and Section 8.1. There, we deal with problems

$$\left\{ \begin{array}{ll} (-\Delta)^s u = f(x, u) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ \frac{u}{w_1} = h & \text{on } \partial\Omega \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} (-\Delta|_{\Omega})^s u = f(x, u) & \text{in } \Omega \\ \frac{u}{w_2} = h & \text{on } \partial\Omega \end{array} \right.$$

In these cases, large s -harmonic functions can be used as sub- or supersolutions whenever the sign of the nonlinearity f allows it. Indeed we solved these nonlinear problems using some sub- and supersolution methods by adapting the arguments in Clément and Sweers [28] and Montenegro and Ponce [67].

Taking, for example, datum $h \equiv 1$ will provide solutions with a boundary explosion of the rate δ^{s-1} and δ^{2s-2} respectively. These solutions are not *large* though, in the following sense. In the classical theory for the Laplacian, large solutions are generated by a compensation between the boundary condition - which tries to push the solution to infinity - and the nonlinearity - which pulls it down, in view of the negative sign. The nonlinearity must provide a sufficiently strong answer to the boundary condition, and in this spirit the Keller-Osserman condition quantifies the necessary rate in order for this phenomenon to take place. Yet so far, in the fractional context, we have actually introduced solutions whose boundary singularity is just due to linear features of the operators. So, at this stage one question is still left unanswered: is it possible to find solutions with a singularity at the boundary generated not by the features of the operator, but rather due to a compensation phenomenon?

¹⁶Here the boundary datum has to be taken as the limit as approaching a point on the boundary.

Large solutions for fractional Laplacian operators. We produce initial answers in Chapters 4 and 8, stating sufficient conditions on the nonlinearity in order for solutions to $(-\Delta)^s u + f(u) = 0$ and $(-\Delta|_\Omega)^s u + f(u) = 0$ to exist, with a singularity at the boundary which is higher order with respect to the ones described in the previous paragraph, so that the singular boundary conditions read as

$$\frac{u}{w_1} = +\infty \text{ on } \partial\Omega, \quad \frac{u}{w_2} = +\infty \text{ on } \partial\Omega$$

which are equivalent to

$$\delta^{1-s}u = +\infty \text{ on } \partial\Omega, \quad \delta^{2-2s}u = +\infty \text{ on } \partial\Omega.$$

The case of the power nonlinearity $f(u) = u^p$ for $(-\Delta)^s$ has been studied by Felmer and Quaas [43] and Chen, Felmer and Quaas [25]. We extend those results by providing explicit bounds on p for the existence, which are

$$p \in \left(1 + 2s, 1 + \frac{2s}{1-s}\right), \quad (17)$$

allowing more general nonlinearities and a clear framework to work in, especially when speaking of boundary conditions. We also prove the following pointwise bound holding throughout the domain

$$\int_{u(x)}^\infty \frac{dt}{\sqrt{F(t)}} \geq c\delta(x)^s, \quad F' = f;$$

this upper bound is computed via an explicit supersolution for the problem. In the particular case where $f(u) = u^p$ this yields

$$u \leq C\delta^{-2s/(p-1)}.$$

In the case of the spectral fractional Laplacian, we study power-like nonlinearities. We deal with

$$\begin{cases} (-\Delta|_\Omega)^s u = -u^p & \text{in } \Omega \\ \delta^{2-2s}u = +\infty & \text{on } \partial\Omega \end{cases}$$

proving the existence of a solution $u \in L^1(\Omega, \delta(x)dx)$ for

$$p \in \left(1 + s, 1 + \frac{s}{1-s}\right). \quad (18)$$

Also in this case we provide the following upper bound, coming from the construction of a supersolution, on the solution

$$u \leq C\delta^{-2s/(p-1)} \quad \text{in } \Omega.$$

This rate coincides with the case of the fractional Laplacian and is sharp in the case $s = 1$.

Let us mention that we suspect (17) and (18) to be optimal, but at present we cannot provide a proof. When p is too large (i.e. it exits the ranges from above), then the candidate supersolution will not fulfil the boundary condition. Moreover, in this case no solution with bounded boundary datum can exist (see Theorem 1.13 and Lemma 4.11 below) so we do not expect the large problem to admit a solution either. Conversely, when p is too small, then we lose the necessary integrability on the candidate supersolution in order to compute its (spectral or not) fractional Laplacian. In both cases we lose some tools necessary for a sub- and supersolution's type method, such as comparison principles, uniqueness of solutions, precise boundary behaviour and the ability to compute sharp subsolutions. These lacks indicate that the theory is just at the beginning and much work can be still done.

When looking at $f(u) = u^p$ both for $(-\Delta)^s$ and $(-\Delta|_\Omega)^s$, a natural question would be if the ranges (17) and (18) of admissible exponents p 's asymptotically converge as $s \uparrow 1$ to the set of admissible exponents for $-\Delta$, which is given by (15) and simply reduces to $p \in (1, +\infty)$. The answer is clearly *no*. While both

$$1 + \frac{2s}{1-s} \rightarrow +\infty \quad \text{and} \quad 1 + \frac{s}{1-s} \rightarrow +\infty \quad \text{as } s \uparrow 1,$$

we only have

$$1 + 2s \rightarrow 3 \quad \text{and} \quad 1 + s \rightarrow 2 \quad \text{as } s \uparrow 1.$$

This is not discouraging, though. In both fractional settings we need respectively that $u \in L^1(\Omega)$ and $u \in L^1(\Omega, \delta(x)dx)$ to make sense of the operator, see for example definitions (2) and (9). This is an additional assumption we do not have in the classical problem, so it is reasonable to get smaller ranges for p . Moreover, the classical solution to the large problem is known to behave like

$$u \asymp \delta^{-2/(p-1)}$$

and such a u is $L^1(\Omega)$ when $p > 3$, while it belongs to $L^1(\Omega, \delta(x)dx)$ for $p > 2$; so, in this sense, we actually have the asymptotic convergence of the admissible ranges of p .

Nonlocal directional curvatures. In the final chapter of the manuscript we introduce a fractional notion of directional curvature for $(N - 1)$ -dimensional smooth manifolds in \mathbb{R}^N . This interest is motivated by the fractional Laplacian context in which several classical problems have been recently rephrased by attracting the attention of a large number of researchers. In particular, in the differential geometry and geometric analysis framework, several new results have been recently obtained for the *diffusion by mean curvature* and the closely related problem of *minimal surfaces* in a nonlocal setting.

As we shall see, the concept of nonlocal mean curvature is naturally associated to a suitable average of appropriate nonlocal directional curvatures, which asymptotically approach their classical counterpart. On the other hand, the nonlocal curvatures seem to have a more messy and fanciful behaviour than the classical ones. In particular we show how, differently from the classical case, minimal and maximal nonlocal directional curvatures are not in general attained at orthogonal directions and the set of extremal directions for nonlocal directional curvatures can be prescribed somehow arbitrarily¹⁷. Some motivations for introducing these new objects can be found in Paragraph 9.2.2.

¹⁷The precise statements of these results are listed in Paragraph 9.2.3.

— Part I —

Boundary blow-up solutions for $(-\Delta)^s$

Overview and main results

In this first Part we study a suitable notion of weak solution to semilinear problems driven by the fractional Laplacian $(-\Delta)^s$, i.e. the integral operator defined as (see e.g. Di Nezza, Palatucci and Valdinoci [37] for an introduction)

$$(-\Delta)^s f(x) = C_{N,s} PV \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x - y|^{N+2s}} dy. \quad (1.1)$$

where $C_{N,s}$ is the normalizing constant defined in (3).

In order to do this, we will need to develop a theory for the Dirichlet problem for the fractional Laplacian with measure data (see Karlsen, Petitta and Ulusoy [55] and Chen and Véron [26], for earlier results in this direction; in particular, look at Paragraph 5.4 for a comparison with [26]). We pay particular attention to those solutions having an explosive behaviour at the boundary of the prescribed domain, known in the literature as *large solutions* or also *boundary blow-up solutions*.

Let us recall that in the classical setting (see Axler, Bourdon and Ramey [5, Theorem 6.9]), to any nonnegative Borel measure $\mu \in \mathcal{M}(\partial B)$ on ∂B it is possible to associate, via the representation through the Poisson kernel, a harmonic function in B with μ as its trace on the boundary. Conversely, any positive harmonic function on the ball B has a trace on ∂B that is a nonnegative Borel measure (see [5, Theorem 6.15]).

When studying the semilinear problem for the Laplacian, solutions can achieve the boundary datum $+\infty$ on the whole boundary. More precisely, take Ω a bounded smooth domain and f nondecreasing such that $f(0) = 0$. According to the works of Keller [56] and Osserman [70], the equation

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega \subseteq \mathbb{R}^N \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} u(x) = +\infty \end{cases}$$

has a solution if and only if f satisfies the so called *Keller-Osserman condition* (15).

In the fractional context, our starting point is that it is possible to provide functions satisfying

$$\begin{cases} (-\Delta)^s u = 0 & \text{in some open bounded region } \Omega \subseteq \mathbb{R}^N, \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} u(x) = +\infty. \end{cases} \quad (1.2)$$

An example of a large s -harmonic function on the unit ball B is

$$u_\sigma(x) = \begin{cases} \frac{c(N,s)}{(1-|x|^2)^\sigma} & \text{in } B \\ \frac{c(N,s+\sigma)}{(|x|^2-1)^\sigma} & \text{in } \mathbb{R}^N \setminus \overline{B} \end{cases} \quad \sigma \in (0, 1-s), \quad c(N,s) = \frac{\Gamma(N/2) \sin(\pi s)}{\pi^{1+N/2}}.$$

See Lemma 2.11 below. Moreover, letting $\sigma \rightarrow 1 - s$ we recover the following example found in Bogdan, Byczkowski, Kulczycki, Ryznar, Song, and Vondraček [13], qualitatively different from the previous one:

$$u_{1-s}(x) = \begin{cases} \frac{c(N, s)}{(1 - |x|^2)^{1-s}} & \text{in } B, \\ 0 & \text{in } \mathbb{R}^N \setminus \overline{B}. \end{cases}$$

The function u_{1-s} so defined satisfies

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } B \\ u = 0 & \text{in } \mathbb{R}^N \setminus \overline{B} \end{cases}$$

and shows how problems where only outer values are prescribed are ill-posed in the classical sense. Different kinds of boundary conditions have to be taken into account: indeed, in the first case we have an s -harmonic function associated to the prescribed data of u_σ outside B ; in the second case all the mass of the boundary datum concentrates on ∂B . This means that we need a notion of weak solution that can deal at the same time with these two different boundary data, one on the complement of the domain and the other one on its boundary. We refer to Grubb [50, e.g. Theorems 4 and 5] for very recent results on the regularity theory for the fractional Laplacian, dealing with this kind of problems.

The notion of s -harmonic function. We start with the definition of s -harmonicity, found in Landkof [60, §6.20]. Denote by

$$\eta(x) = \begin{cases} \frac{c(N, s)}{|x|^N (|x|^2 - 1)^s} & |x| > 1 \\ 0 & |x| \leq 1 \end{cases}$$

and by

$$\eta_r(x) = \frac{1}{r^N} \eta\left(\frac{x}{r}\right) = \begin{cases} \frac{c(N, s) r^{2s}}{|x|^N (|x|^2 - r^2)^s} & |x| > r, \\ 0 & |x| \leq r. \end{cases}$$

The constant $c(N, s)$ above is chosen in such a way that

$$\int_{\mathbb{R}^N} \eta(x) dx = \int_{\mathbb{R}^N} \eta_r(x) dx = 1 \quad (1.3)$$

and therefore, see [60, §6.19]

$$c(N, s) = \frac{\Gamma(N/2) \sin(\pi s)}{\pi^{1+N/2}}. \quad (1.4)$$

The definition of s -harmonicity is given via a mean value property, namely

DEFINITION 1.1: s -HARMONIC FUNCTION | We say that a measurable nonnegative function $u : \mathbb{R}^N \rightarrow [0, +\infty]$ is s -harmonic on an open set $\Omega \subseteq \mathbb{R}^N$ if $u \in C(\Omega)$ and for any $x \in \Omega$ and $0 < r < \text{dist}(x, \partial\Omega)$

$$u(x) = \int_{\mathbb{R}^N \setminus B_r(x)} \frac{c(N, s) r^{2s}}{|y - x|^N (|y - x|^2 - r^2)^s} u(y) dy = (\eta_r * u)(x). \quad (1.5)$$

1.1 AN INTEGRATION BY PARTS FORMULA

For any two functions u, v in the Schwartz class \mathcal{S} , the self-adjointness of the operator $(-\Delta)^s$ entails

$$\int_{\mathbb{R}^N} u (-\Delta)^s v = \int_{\mathbb{R}^N} v (-\Delta)^s u,$$

this follows from the representation of the fractional Laplacian via the Fourier transform, see [37, Paragraph 3.1]. By splitting \mathbb{R}^N into two domains of integration

$$\int_{\Omega} u(-\Delta)^s v - \int_{\Omega} v(-\Delta)^s u = \int_{\mathbb{R}^N \setminus \Omega} v(-\Delta)^s u - \int_{\mathbb{R}^N \setminus \Omega} u(-\Delta)^s v. \quad (1.6)$$

Denote by

$$C^{2s+\varepsilon}(\Omega) = \{v : \mathbb{R}^N \rightarrow \mathbb{R} \text{ such that } v \in C(\Omega) \text{ and for any } K \subset \subset \Omega \text{ there exists } \alpha = \alpha(K, v) \text{ such that } v \in C^{2s+\alpha}(K)\}.$$

PROPOSITION 1.2 | *Let $\Omega \subseteq \mathbb{R}^N$ open and bounded. If $u \in C^{2s+\varepsilon}(\Omega) \cap L^\infty(\mathbb{R}^N)$ and $v = 0$ in $\mathbb{R}^N \setminus \Omega$, $v \in C^{2s+\varepsilon}(\Omega) \cap C^s(\mathbb{R}^N)$ and $(-\Delta)^s v \in L^1(\mathbb{R}^N)$, then*

$$\int_{\Omega} u(-\Delta)^s v - \int_{\Omega} v(-\Delta)^s u = - \int_{\mathbb{R}^N \setminus \Omega} u(-\Delta)^s v. \quad (1.7)$$

The proof can be found in Paragraph 2.2.1.

From now on the set $\Omega \subseteq \mathbb{R}^N$ will be an open bounded domain with $C^{1,1}$ boundary. More generally, we will prove in Section 2.2 the following

PROPOSITION 1.3 | *Let $\delta(x) = \text{dist}(x, \partial\Omega)$ for any $x \in \mathbb{R}^N$, and $u \in C^{2s+\varepsilon}(\Omega)$ such that*

$$\delta(x)^{1-s} u(x) \in C(\overline{\Omega}) \quad \text{and} \quad \int_{\mathcal{C}\Omega} \frac{|u(x)|}{1 + |x|^{N+2s}} \cdot \frac{dx}{\delta(x)^s} < +\infty. \quad (1.8)$$

Let

a) $G_\Omega(x, y)$, $x, y \in \Omega$, $x \neq y$, be the Green function of the fractional Laplacian on Ω , that is

$$G_\Omega(x, y) = \frac{-C_{N,-s}}{|x - y|^{N-2s}} - H_\Omega(x, y)$$

and H_Ω is the unique function in $C^s(\mathbb{R}^N)$ solving

$$\begin{cases} (-\Delta)^s H_\Omega(x, \cdot) = 0 & \text{in } \Omega \\ H_\Omega(x, y) = \frac{-C_{N,-s}}{|x - y|^{N-2s}} & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \end{cases}$$

pointwisely,¹

b) $P_\Omega(x, y) = -(-\Delta)^s G_\Omega(x, y)$, $x \in \Omega, y \in \mathbb{R}^N \setminus \overline{\Omega}$, be the corresponding Poisson kernel (where the fractional Laplacian is computed in the variable $y \in \mathcal{C}\Omega$ and treating $x \in \Omega$ as a fixed parameter),

c) for $x \in \Omega, \theta \in \partial\Omega$,²

$$M_\Omega(x, \theta) = \lim_{\substack{y \rightarrow \theta \\ y \in \Omega}} \frac{G_\Omega(x, y)}{\delta(y)^s},$$

d) for $\theta \in \partial\Omega$,

$$Eu(\theta) = \lim_{\substack{x \rightarrow \theta \\ x \in \Omega}} \frac{u(x)}{\int_{\partial\Omega} M_\Omega(x, \theta') d\mathcal{H}(\theta')},$$

where the limit is well-defined in view of Lemma 2.7 below.

¹The construction of H can be found in the proof of Theorem 2.1 below.

²This is a readaptation of the Martin kernel of Ω .

Then the integration by parts formula

$$\int_{\Omega} u (-\Delta)^s v = \int_{\Omega} (-\Delta)^s u v - \int_{\mathbb{R}^N \setminus \overline{\Omega}} u (-\Delta)^s v + \int_{\partial\Omega} Eu(\theta) D_s v(\theta) d\mathcal{H}(\theta) \quad (1.9)$$

holds, where

$$D_s v(\theta) = \lim_{\substack{x \rightarrow \theta \\ x \in \Omega}} \frac{v(x)}{\delta(x)^s}, \quad \theta \in \partial\Omega$$

for any $v \in C^s(\mathbb{R}^N)$ such that $(-\Delta)^s v|_{\Omega} \in C_c^\infty(\Omega)$ and $v \equiv 0$ in $\mathbb{R}^N \setminus \Omega$. Such a limit exists and is continuous in θ in view of Lemma 2.12. In addition, we have the representation formula

$$u(x) = \int_{\Omega} G_{\Omega}(x, y) (-\Delta)^s u(y) dy - \int_{\mathbb{R}^N \setminus \overline{\Omega}} (-\Delta)^s G_{\Omega}(x, y) u(y) dy + \int_{\partial\Omega} D_s G_{\Omega}(x, \theta) Eu(\theta) d\mathcal{H}(\theta). \quad (1.10)$$

The present Part is organized as follows. Section 2.1 relates Definition 1.1 with the fractional Laplacian $(-\Delta)^s$. Section 2.2 recalls some facts on Green functions and Poisson kernels and it studies the linear Dirichlet problem both in the pointwise and in a weak sense. Chapter 3 deals with the nonlinear problem. Now, let us outline the main results in Section 2.2 and Chapter 3.

1.2 THE DIRICHLET PROBLEM

For a fixed $x \in \Omega$ the Poisson kernel satisfies

$$0 \leq -(-\Delta)^s G_{\Omega}(x, y) \leq c \min \left\{ \delta(y)^{-s}, \delta(y)^{-N-2s} \right\}, \quad y \in \mathbb{R}^N \setminus \overline{\Omega}$$

for some constant $c > 0$ independent of x and y , as a consequence of (2.11) and (2.19) below. In particular any Dirichlet condition $u = g$ in $\mathbb{R}^N \setminus \overline{\Omega}$ satisfying (1.11) below is admissible in the representation formula (1.10). We prove the following

THEOREM 1.4 | Let $f \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1)$, $g : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow \mathbb{R}$ be any measurable function satisfying³

$$\int_{\mathbb{R}^N \setminus \overline{\Omega}} |g(y)| \min \left\{ \delta(y)^{-s}, \delta(y)^{-N-2s} \right\} dy < +\infty \quad (1.11)$$

and $h \in C(\partial\Omega)$. Then, the function defined by setting

$$u(x) = \begin{cases} \int_{\Omega} f(y) G_{\Omega}(y, x) dy - \int_{\mathbb{R}^N \setminus \overline{\Omega}} g(y) (-\Delta)^s G_{\Omega}(x, y) dy + \\ \quad + \int_{\partial\Omega} h(\theta) M_{\Omega}(x, \theta) d\mathcal{H}(\theta), & x \in \Omega \\ g(x), & x \in \mathbb{R}^N \setminus \overline{\Omega} \end{cases} \quad (1.12)$$

belongs to $C^{2s+\varepsilon}(\Omega)$, fulfils (1.8), and u is the only pointwise solution of

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \\ Eu = h & \text{on } \partial\Omega. \end{cases}$$

Here Eu has been defined in Proposition 1.3. Moreover, if $g \in C(\overline{V_\varepsilon})$ for some $\varepsilon > 0$, where $V_\varepsilon = \{x \in \mathbb{R}^N \setminus \overline{\Omega} : \delta(x) < \varepsilon\}$ and $h = 0$, then $u \in C(\overline{\Omega})$.

REMARK 1.5 | Even if it is irrelevant to write $\int_{\mathbb{R}^N \setminus \Omega}$ or $\int_{\mathbb{R}^N \setminus \overline{\Omega}}$ in formula (1.12), with the latter notation we would like to stress on the fact that the boundary $\partial\Omega$ plays an important role in this setting.

³Compare also with equation (2.18) below.

REMARK 1.6 | Since $\partial\Omega \in C^{1,1}$, we will exploit the behaviour of the Green function and the Poisson kernel described by [27, Theorem 2.10 and equation (2.13) resp.]: there exist $c_1 = c_1(\Omega, s) > 0$, $c_2 = c_2(\Omega, s)$ such that

$$\begin{aligned} \frac{\delta(x)^s}{c_1 \delta(y)^s (1 + \delta(y))^s |x - y|^N} &\leq -(-\Delta)^s G_\Omega(x, y) \leq \\ &\leq \frac{c_1 \delta(x)^s}{\delta(y)^s (1 + \delta(y))^s |x - y|^N}, \end{aligned} \quad x \in \Omega, y \in \mathbb{R}^N \setminus \overline{\Omega}, \quad (1.13)$$

and

$$\begin{aligned} \frac{1}{c_2 |x - y|^N} (|x - y|^2 \wedge \delta(x)\delta(y))^s &\leq G_\Omega(x, y) \leq \\ &\leq \frac{c_2}{|x - y|^N} (|x - y|^2 \wedge \delta(x)\delta(y))^s \end{aligned} \quad x, y \in \Omega. \quad (1.14)$$

REMARK 1.7: CONSTRUCTION OF LARGE s -HARMONIC FUNCTIONS | The case when $f = 0$ in Theorem 1.4 corresponds to s -harmonic functions: when $h \not\equiv 0$ then u automatically explodes somewhere on the boundary (by definition of E), while if $h \equiv 0$ then large s -harmonic functions can be built as follows. Take any positive g satisfying (1.11) with

$$\lim_{\delta(x) \downarrow 0} g(x) = +\infty$$

and let $g_n = \min\{g, n\}$, $n \in \mathbb{N}$. By Theorem 1.4, the corresponding solutions $u_n \in C(\overline{\Omega})$. In particular, $u_n = n$ on $\partial\Omega$ and by the Maximum Principle $\{u_n\}_{n \in \mathbb{N}}$ is increasing. Hypothesis (1.11) guarantees a uniform pointwise bound far from $\partial\Omega$ on $\{u_n\}_{n \in \mathbb{N}}$. Then

$$u_n(x) = - \int_{\mathbb{R}^N \setminus \Omega} g_n(y) (-\Delta)^s G_\Omega(x, y) dy$$

increases to a s -harmonic function u such that

$$\liminf_{\substack{x \in \Omega \\ x \rightarrow \partial\Omega}} u(x) \geq \liminf_{\substack{x \in \Omega \\ x \rightarrow \partial\Omega}} u_n(x) = n, \quad \text{for any } n \in \mathbb{N}.$$

Next, in view of Theorem 1.4, we introduce the test function space

$$\mathcal{T}(\Omega) = \{\phi \in C^\infty(\Omega) : \text{there exists } \psi \in C_c^\infty(\Omega) \text{ such that} \\ (-\Delta)^s \phi = \psi \text{ in } \Omega, \phi = 0 \text{ in } \mathbb{R}^N \setminus \overline{\Omega}, E\phi = 0 \text{ on } \partial\Omega\}.$$

Note that $\mathcal{T}(\Omega) \subseteq C^s(\mathbb{R}^N)$, by [73, Proposition 1.1]. Starting from the integration by parts formula (1.9), we introduce the following notion of weak solution

DEFINITION 1.8: WEAK L^1 SOLUTION | Given three Radon measures $\lambda \in \mathcal{M}(\Omega)$, $\mu \in \mathcal{M}(\mathbb{R}^N \setminus \Omega)$ and $\nu \in \mathcal{M}(\partial\Omega)$, such that

$$\int_{\Omega} \delta(x)^s d|\lambda|(x) < +\infty, \int_{\mathbb{R}^N \setminus \overline{\Omega}} \min\{\delta(x)^{-s}, \delta(x)^{-N-2s}\} d|\mu|(x) < +\infty, |\nu|(\Omega) < +\infty$$

we say that a function $u \in L^1(\Omega)$ is a solution of

$$\begin{cases} (-\Delta)^s u = \lambda & \text{in } \Omega \\ u = \mu & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = \nu & \text{on } \partial\Omega \end{cases}$$

if for every $\phi \in \mathcal{T}(\Omega)$ it holds

$$\int_{\Omega} u(x) (-\Delta)^s \phi(x) dx = \int_{\Omega} \phi(x) d\lambda(x) - \int_{\mathbb{R}^N \setminus \overline{\Omega}} (-\Delta)^s \phi(x) d\mu(x) + \int_{\partial\Omega} D_s \phi(\theta) d\nu(\theta). \quad (1.15)$$

The integrals in the definition are finite for any $\phi \in \mathcal{T}(\Omega)$ in view of Lemma 2.12.

Thanks to the representation formula (1.10), we prove

THEOREM 1.9 | *Given two Radon measures $\lambda \in \mathcal{M}(\Omega)$ and $\mu \in \mathcal{M}(\mathbb{R}^N \setminus \overline{\Omega})$ such that*

$$\int_{\Omega} G_{\Omega}(x, y) d|\lambda|(y) < +\infty, \quad \text{and} \quad - \int_{\mathbb{R}^N \setminus \overline{\Omega}} (-\Delta)^s G_{\Omega}(x, y) d|\mu|(y) < +\infty,$$

for a.e. $x \in \Omega$, and a Radon measure $\nu \in \mathcal{M}(\partial\Omega)$ such that $|\nu|(\partial\Omega) < +\infty$, the problem

$$\begin{cases} (-\Delta)^s u = \lambda & \text{in } \Omega \\ u = \mu & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = \nu & \text{on } \partial\Omega \end{cases}$$

admits a unique solution $u \in L^1(\Omega)$ in the weak sense. In addition, we have the representation formula

$$u(x) = \int_{\Omega} G_{\Omega}(x, y) d\lambda(y) - \int_{\mathbb{R}^N \setminus \overline{\Omega}} (-\Delta)^s G_{\Omega}(x, y) d\mu(y) + \int_{\partial\Omega} M_{\Omega}(x, y) d\nu(y)$$

and

$$\|u\|_{L^1(\Omega)} \leq C \left(\|\delta(x)^s \lambda\|_{\mathcal{M}(\Omega)} + \|\min\{\delta(x)^{-s}, \delta(x)^{-N-2s}\} \mu\|_{\mathcal{M}(\mathbb{R}^N \setminus \overline{\Omega})} + \|\nu\|_{\mathcal{M}(\partial\Omega)} \right)$$

for some constant $C = C(n, s, \Omega) > 0$.

We will conclude Section 2.2 by showing that

PROPOSITION 1.10 | *The weak solution of*

$$\begin{cases} (-\Delta)^s u(x) = \frac{1}{\delta(x)^{\beta}} & \text{in } \Omega, \beta \in (0, 1+s) \\ u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies

$$\begin{aligned} c_1 \delta(x)^s &\leq u(x) \leq c_2 \delta(x)^s && \text{for } 0 < \beta < s, \\ c_3 \delta(x)^s \log \frac{1}{\delta(x)} &\leq u(x) \leq c_4 \delta(x)^s \log \frac{1}{\delta(x)} && \text{for } \beta = s, \\ c_5 \delta(x)^{-\beta+2s} &\leq u(x) \leq c_6 \delta(x)^{-\beta+2s} && \text{for } s < \beta < 1+s. \end{aligned} \tag{1.16}$$

Moreover, there exist constants $\underline{c} = \underline{c}(n, s, \Omega) > 0$ and $\overline{c} = \overline{c}(n, s, \Omega) > 0$ such that the solution of

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies

$$\underline{c} \underline{g}(\delta(x)) := \underline{c} \inf_{\delta(y)=\delta(x)} g(y) \leq u(x) \leq \overline{c} \sup_{\delta(y)=\delta(x)} g(y) =: \overline{c} \overline{g}(\delta(x)) \quad x \in \Omega, \tag{1.17}$$

for any g satisfying (1.11) such that $\underline{g}, \overline{g}$ are decreasing functions in 0^+ with $\underline{g} \leq \overline{g}$ and

$$\lim_{t \downarrow 0} \underline{g}(t) = +\infty.$$

1.3 SEMILINEAR DIRICHLET PROBLEMS

In the following, $\Omega \subset \mathbb{R}^N$ will always be a bounded open region with $C^{1,1}$ boundary. We consider nonlinearities $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying hypotheses

f.1) $f \in C(\Omega \times \mathbb{R})$, $f \in L^\infty(\Omega \times I)$ for any bounded $I \subseteq \mathbb{R}$

f.2) $f(x, 0) = 0$ for any $x \in \Omega$, and $f(x, t) \geq 0$ for any $x \in \Omega$, $t > 0$,

and all positive boundary data g that satisfy (1.11).

After having constructed large s -harmonic functions, we first prove the following preliminary

THEOREM 1.11 | *Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying f.1). Let $g : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow \mathbb{R}$ be a measurable bounded function. Assume the nonlinear problem*

$$\begin{cases} (-\Delta)^s u = -f(x, u) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

admits a subsolution $\underline{u} \in L^1(\Omega)$ and a supersolution $\overline{u} \in L^1(\Omega)$ in the weak sense

$$\begin{cases} (-\Delta)^s \underline{u} \leq -f(x, \underline{u}) & \text{in } \Omega \\ \underline{u} \leq g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ E\underline{u} = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta)^s \overline{u} \geq -f(x, \overline{u}) & \text{in } \Omega \\ \overline{u} \geq g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ E\overline{u} = 0 & \text{on } \partial\Omega \end{cases}$$

i.e. for any $\phi \in \mathcal{T}(\Omega)$, $(-\Delta)^s \phi|_\Omega \geq 0$, it holds

$$\int_\Omega \underline{u} (-\Delta)^s \phi \leq - \int_\Omega f(x, \underline{u}) \phi - \int_{\mathbb{R}^N \setminus \overline{\Omega}} g (-\Delta)^s \phi, \quad \int_\Omega \overline{u} (-\Delta)^s \phi \geq - \int_\Omega f(x, \overline{u}) \phi - \int_{\mathbb{R}^N \setminus \overline{\Omega}} g (-\Delta)^s \phi.$$

Assume also $\underline{u} \leq \overline{u}$ in Ω , and $\underline{u}, \overline{u} \in L^\infty(\Omega) \cap C(\Omega)$. Then the above nonlinear problem has a weak solution u in the sense of Definition 1.8 satisfying

$$\underline{u} \leq u \leq \overline{u}.$$

In addition,

- *if f is increasing in the second variable, i.e. $f(x, s) \leq f(x, t)$ whenever $s \leq t$, for all $x \in \Omega$, then there is a unique solution,*
- *if not, there is a unique minimal solution u_1 , that is a solution u_1 such that $\underline{u} \leq u_1 \leq v$ for any other supersolution $v \geq \underline{u}$.*

In case our boundary datum g is a nonnegative bounded function, then Theorem 1.11 (with f.2) as an additional assumption) provides a unique solution since we may consider $\overline{u} = \sup g$ and $\underline{u} = 0$. Then we attack directly the problem with unbounded boundary values, and we are especially interested in those data exploding on $\partial\Omega$. The existence of large s -harmonic functions turns out to be the key ingredient to prove all the following theorems, that is,

THEOREM 1.12: CONSTRUCTION OF LARGE SOLUTIONS | *Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying f.1) and f.2). Then there exist $u, v : \mathbb{R}^N \rightarrow [0, +\infty]$ solving*

$$\begin{cases} (-\Delta)^s u = -f(x, u) & \text{in } \Omega \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} u(x) = +\infty \end{cases} \quad \text{and} \quad \begin{cases} (-\Delta)^s v = f(x, v) & \text{in } \Omega \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} v(x) = +\infty, \end{cases}$$

in a weak L^1 sense.

Depending on the nature of the nonlinearity f one can be more precise about the Dirichlet values of u . Namely,

THEOREM 1.13 | *Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying f.1) and f.2), and $g : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow [0, +\infty]$ a measurable function satisfying (1.11); let also be $h \in C(\partial\Omega)$, $h \geq 0$. The semilinear problem*

$$\begin{cases} (-\Delta)^s u(x) = -f(x, u(x)) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = h & \text{on } \partial\Omega \end{cases}$$

satisfies the following:

i) *if $h \equiv 0$, the equation has a weak solution for any g satisfying (1.11),*

ii) *if $h \not\equiv 0$ then*

- *the problem has a weak solution if there exist a nondecreasing function $\Lambda : (0, +\infty) \rightarrow (0, \infty)$ and $a > 0$ such that*

$$f(x, t) \leq \Lambda(at), \quad \text{for } t > 0$$

$$\text{and } \Lambda(\delta^{s-1})\delta^s \in L^1(\Omega),$$

- *the problem does not admit any weak solution if there exist $b_1, T > 0$ such that*

$$f(x, t) \geq b_1 t^{\frac{1+s}{1-s}}, \quad \text{for } t > T.$$

If, in addition, f is increasing in the second variable then the problem admits only one positive solution.

REMARK 1.14 | To our knowledge, the only previous results dealing with semilinear problems driven by the fractional Laplacian are [43] and [25], as mentioned in the Introduction. On the one hand the above theorem improves them in the following ways:

1. we have quite general assumptions on the nonlinearity, while in [43] and [25] only power-like nonlinearities are taken into account,
2. we give the exact value of the threshold growth (i.e. $f(t) \sim t^{(1+s)/(1-s)}$) distinguishing the existence and the nonexistence of solutions,
3. we specify in what sense solutions blow up at the boundary via the E trace operator, clarifying why the authors in [43] and [25] find infinitely many solutions.

On the other hand we do not find *all* boundary blow-up solutions which are proved to exist in [43] and [25]: these correspond to boundary data $h \equiv +\infty$ or $g \in L^1(\mathbb{R}^N \setminus \Omega)$ not satisfying (1.11). This last question will be studied in the following.

THEOREM 1.15: SUBLINEAR SOURCE | *Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying f.1) and f.2), and $g : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow [0, +\infty]$ a measurable function satisfying (1.11); let also be $h \in C(\partial\Omega)$, $h \geq 0$. Suppose also that*

$$f(x, t) \leq \Lambda(t), \quad \text{for all } x \in \overline{\Omega}, t \geq 0$$

where $\Lambda \in C^1(0, +\infty)$ is concave and $\Lambda'(t) \xrightarrow{t \uparrow \infty} 0$. Then there exists a positive weak solution u to the semilinear problem

$$\begin{cases} (-\Delta)^s u(x) = f(x, u(x)) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = h & \text{on } \partial\Omega. \end{cases}$$

THEOREM 1.16: SUPERLINEAR SOURCE | Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying f.1) and f.2). For $0 < \beta < 1 - s$, consider problems

$$\begin{aligned}
 (\star) \quad & \begin{cases} (-\Delta)^s u(x) = \lambda f(x, u(x)) & \text{in } \Omega \\ u(x) = \delta(x)^{-\beta} & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = 0 & \text{on } \partial\Omega \end{cases} \\
 (\star\star) \quad & \begin{cases} (-\Delta)^s u(x) = \lambda f(x, u(x)) & \text{in } \Omega \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = 1 & \text{on } \partial\Omega. \end{cases}
 \end{aligned}$$

Existence. If there exist $a_1, a_2, T > 0$ and $p \geq 1$, such that

$$f(x, t) \leq a_1 + a_2 t^p, \quad x \in \Omega, \quad t > T$$

and $p\beta < 1 + s$, then there exists $L_1 > 0$ depending on β and p such that problem (\star) admits a weak solution $u \in L^1(\Omega)$ for any $\lambda \in [0, L_1]$. Similarly, if $p(1 - s) < 1 + s$, then there exists $L_2 > 0$ depending on p such that problem $(\star\star)$ admits a weak solution $u \in L^1(\Omega)$ for any $\lambda \in [0, L_2]$.

Nonexistence. If there exist $b, T > 0$ and $q > 0$, such that

$$bt^q \leq f(x, t), \quad x \in \Omega, \quad t > T$$

and $q\beta \geq 1 + s$, then problem (\star) admits a weak solution only for $\lambda = 0$. Similarly, if $q(1 - s) \geq 1 + s$, then problem $(\star\star)$ admits a weak solution only for $\lambda = 0$.

We finally note that, with the definition of weak solution we are dealing with, the nonexistence of a weak solution implies *complete blow-up*, meaning that:

DEFINITION 1.17: COMPLETE BLOW-UP | If for any nondecreasing sequence $\{f_k\}_{k \in \mathbb{N}}$ of bounded functions such that $f_k \uparrow f$ pointwisely as $k \uparrow +\infty$, and any sequence $\{u_k\}_{k \in \mathbb{N}}$ of positive solutions to

$$\begin{cases} (-\Delta)^s u_k = f_k(x, u_k) & \text{in } \Omega \\ u_k = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu_k = h & \text{on } \partial\Omega, \end{cases}$$

there holds

$$\lim_{k \uparrow +\infty} \frac{u_k(x)}{\delta(x)^s} = +\infty, \quad \text{uniformly in } x \in \Omega,$$

then we say there is complete blow-up.

THEOREM 1.18 | Let $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying f.1) and f.2) and $g : \mathbb{R}^N \setminus \overline{\Omega} \rightarrow [0, +\infty]$ a measurable function satisfying (1.11); let also be $h \in C(\partial\Omega)$, $h \geq 0$. If there is no weak solution to

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \\ Eu = h & \text{on } \partial\Omega, \end{cases}$$

then there is complete blow-up.

1.3.1 The interest in larger solutions

So, we have achieved the existence of boundary blow-up solutions to nonlinear problems with negative right-hand side, provided by Theorem 1.13; anyhow this singular behaviour is driven by a linear phenomenon rather than a compensation between the nonlinearity and the explosion (as in the classical case), indeed no growth condition on f arises except when $h \not\equiv 0$, where one essentially needs

$$\int_{\Omega} f(x, \delta(x)^{s-1}) \delta(x)^s dx < \infty$$

in order to make sense of the weak L^1 definition.

For this reason in what follows we address the question of the existence of solutions to problems of the form

$$\begin{cases} (-\Delta)^s u = -f(u) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \Omega \\ \delta^{1-s} u = +\infty & \text{on } \partial\Omega \end{cases} \quad g \geq 0, \quad \int_{\mathbb{R}^N \setminus \Omega} g \delta^{-s} \min\{1, \delta^{-N-s}\} = +\infty.$$

providing sufficient conditions for their solvability. In doing so, we start answering the question left opened in Remark 1.14. The results listed in Theorems 1.21 and 1.23 below can be applied for a particular case of the *fractional singular Yamabe problem*, see e.g. González, Mazzeo and Sire [49].

1.4 TOWARDS A FRACTIONAL KELLER-OSSERMAN CONDITION

In Chapter 4 we will work in the following set of assumptions:

- Ω is a bounded open domain of class C^2 ,
- $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$f \text{ is an increasing } C^1 \text{ function with } f(0) = 0 \quad (1.18)$$

- F is the antiderivative of f vanishing in 0:

$$F(t) := \int_0^t f(\tau) d\tau, \quad (1.19)$$

- there exist $0 < m < M$, such that

$$1 + m \leq \frac{tf'(t)}{f(t)} \leq 1 + M, \quad (1.20)$$

and thus f satisfies (15) because, integrating the lower inequality, one gets

$$f(t) \geq f(1)t^{1+m} \quad \text{and} \quad F(t) \geq \frac{f(1)}{2+m} t^{2+m},$$

we can therefore define the function

$$\phi(u) := \int_u^{+\infty} \frac{dt}{\sqrt{F(t)}}, \quad (1.21)$$

- f satisfies

$$\int_1^{+\infty} \phi(t)^{1/s} dt < +\infty. \quad (1.22)$$

In what follows we will use the expression $g \asymp h$ where $g, h : (0, +\infty) \rightarrow (0, +\infty)$ to shorten

there exists $C > 0$ such that $\frac{h(t)}{C} \leq g(t) \leq Ch(t)$, for any $t > 0$.

REMARK 1.19 | The function $\phi : (0, +\infty) \rightarrow (0, +\infty)$ is monotone decreasing and

$$\lim_{t \downarrow 0} \phi(t) = +\infty, \quad \lim_{t \uparrow +\infty} \phi(t) = 0.$$

Moreover

$$\phi'(u) = -\frac{1}{\sqrt{F(u)}}.$$

is of the same order as $-(u f(u))^{-1/2}$ since for $t > 0$ and some $\tau \in (0, t)$

$$\frac{F(t)}{t f(t)} = \frac{f(\tau)}{f(\tau) + \tau f'(\tau)} \begin{cases} \geq \frac{1}{2+M} \\ \leq \frac{1}{2+m} \end{cases} \quad \text{by the Cauchy Theorem.}$$

This entails that the order of $\phi(u)$ is the same as $(u/f(u))^{1/2}$ indeed for $u > 0$ and some $t \in (u, +\infty)$

$$\frac{\sqrt{\frac{u}{f(u)}}}{\phi(u)} = \frac{\frac{1}{2} \sqrt{\frac{f(t)}{t}} \cdot \frac{f(t)-t f'(t)}{f(t)^2}}{\phi'(t)} \asymp \frac{f(t)-t f'(t)}{-f(t)} = \frac{t f'(t)}{f(t)} - 1$$

which belongs to (m, M) by hypothesis (1.20). Note that hypothesis (1.22) is therefore equivalent to

$$\int_1^{+\infty} \left(\frac{t}{f(t)} \right)^{\frac{1}{2s}} dt < +\infty. \quad (1.23)$$

REMARK 1.20 | In [56] and [70] condition (15) is proven to be necessary and sufficient for the existence of a solution of

$$\begin{cases} -\Delta u = -f(u) & \text{in } \Omega, \\ \lim_{x \rightarrow \partial\Omega} u(x) = +\infty. \end{cases}$$

Note that if we set $s = 1$ in (1.22) then

$$+\infty > \int_u^{+\infty} \phi(t) dt \asymp \int_u^{+\infty} \sqrt{\frac{t}{f(t)}} dt \asymp \int_u^{+\infty} \frac{t}{\sqrt{F(t)}} dt$$

we get the condition to force the classical solution u to be $L^1(\Omega)$. Indeed in [40, Theorem 1.6] it is proved that a solution u satisfies

$$\lim_{x \rightarrow \partial\Omega} \frac{\phi(u(x))}{\delta(x)} = 1 \quad (1.24)$$

which yields that $u \in L^1(\Omega)$ if and only if ϕ^{-1} , the inverse function of ϕ (recall it is monotone decreasing), is integrable in a neighbourhood of 0, i.e. with a change of integration variable

$$+\infty > \int_0^\eta \phi^{-1}(r) dr = \int_{\phi^{-1}(\eta)}^{+\infty} t |\phi'(t)| dt = \int_{t_0}^{+\infty} \frac{t}{\sqrt{F(t)}} dt.$$

Our results can be summarized as follows.

THEOREM 1.21 | Suppose that the nonlinear term f satisfies hypotheses (1.18), (1.20), (1.22) and

$$\int_{t_0}^{+\infty} f(t) t^{-2/(1-s)} dt < +\infty. \quad (1.25)$$

Then problem

$$\begin{cases} (-\Delta)^s u = -f(u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \\ \delta^{1-s} u = +\infty & \text{on } \partial\Omega \end{cases} \quad (1.26)$$

admits a solution $u \in L^1(\Omega)$. Moreover there exists $c > 0$ for which

$$\phi(u(x)) \geq c \delta(x)^s \quad \text{near } \partial\Omega. \quad (1.27)$$

REMARK 1.22 | The condition $u \in L^1(\Omega)$ is necessary to make sense of the fractional Laplacian, as any different possible definition points out (see the Introduction). Also, compare the boundary behaviour in this setting expressed by equation (1.27), with the classical one in equation (1.24).

THEOREM 1.23 | Suppose that the nonlinear term f satisfies hypotheses (1.18), (1.20), (1.22) and

$$\begin{aligned} g : \mathbb{R}^N \setminus \Omega &\longrightarrow [0, +\infty), & g &\in L^1(\mathbb{R}^N \setminus \Omega) \\ \phi(g(x)) &\geq \delta(x)^s, & &\text{near } \partial\Omega. \end{aligned} \quad (1.28)$$

Then problem

$$\begin{cases} (-\Delta)^s u = -f(u) & \text{in } \Omega \\ u = g & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \quad (1.29)$$

admits a solution $u \in L^1(\Omega)$. Moreover there exists $c > 0$ for which

$$\phi(u(x)) \geq c \delta(x)^s \quad \text{near } \partial\Omega.$$

Mind that in problem (1.29) we do not prescribe the singular trace at $\partial\Omega$.

Notations

In the following we will always use the following notations:

- $\mathcal{C}\Omega$, when $\Omega \subseteq \mathbb{R}^N$ is open, for $\mathbb{R}^N \setminus \overline{\Omega}$,
- $\delta(x)$ for $\text{dist}(x, \partial\Omega)$ once $\Omega \subseteq \mathbb{R}^N$ has been fixed,
- $\mathcal{M}(\Omega)$, when $\Omega \subseteq \mathbb{R}^N$, for the space of measures on Ω ,
- \mathcal{H} , for the $n - 1$ dimensional Hausdorff measure, dropping the “ $n - 1$ ” subscript whenever there is no ambiguity,
- $f \wedge g$, when f, g are two functions, for the function $\min\{f, g\}$,
- $C^{2s+\varepsilon}(\Omega) = \{v : \mathbb{R}^N \rightarrow \mathbb{R} \text{ and for any } K \text{ compactly supported in } \Omega, \text{ there exists } \alpha = \alpha(K, v) \text{ such that } v \in C^{2s+\alpha}(K)\}$.

Linear fractional Dirichlet problems

2.1 A MEAN VALUE FORMULA

Definition 1.1 of s -harmonicity turns out to be equivalent to have a null fractional Laplacian. Since we could not find a precise reference for this fact, we provide a proof. Indeed, on the one hand we have that any function u which is s -harmonic in an open set Ω solves

$$(-\Delta)^s u(x) = 0 \quad \text{in } \Omega,$$

indeed condition (1.5) can be rewritten, using (1.4),

$$\int_{CB_r(x)} \frac{u(y) - u(x)}{|y - x|^N (|y - x|^2 - r^2)^s} dy = 0 \quad \text{for any } r \in (0, \text{dist}(x, \partial\Omega))$$

and therefore

$$\begin{aligned} 0 &= \lim_{r \downarrow 0} \int_{CB_r(x)} \frac{u(y) - u(x)}{|y - x|^N (|y - x|^2 - r^2)^s} dy \\ &= PV \int_{\mathbb{R}^N} \frac{u(y) - u(x)}{|y - x|^{N+2s}} dy = -\frac{(-\Delta)^s u(x)}{C_{N,s}}. \end{aligned}$$

Indeed, by dominated convergence, far from x it holds

$$\int_{CB_1(x)} \frac{u(y) - u(x)}{|y - x|^N (|y - x|^2 - r^2)^s} dy \xrightarrow{r \downarrow 0} \int_{CB_1(x)} \frac{u(y) - u(x)}{|y - x|^{N+2s}} dy.$$

Now, any function u s -harmonic in Ω is smooth in Ω : this follows from the representation through the Poisson kernel on balls, given in Theorem 1.4, and the smoothness of the Poisson kernel, see formula (2.21) below. Since u is a smooth function, a Taylor expansion when $|y - x| < 1$

$$u(y) - u(x) = \langle \nabla u(x), y - x \rangle + \theta(y - x), \quad \text{where } |\theta(y - x)| \leq C |y - x|^2$$

implies

$$\begin{aligned} & \left| \int_{B_1(x) \setminus B_r(x)} \frac{u(y) - u(x)}{|y - x|^N (|y - x|^2 - r^2)^s} dy - \int_{B_1(x) \setminus B_r(x)} \frac{u(y) - u(x)}{|y - x|^{N+2s}} dy \right| \\ & \leq \int_{B_1(x) \setminus B_r(x)} \frac{|\theta(y - x)|}{|y - x|^N} \left(\frac{1}{(|y - x|^2 - r^2)^s} - \frac{1}{|y - x|^{2s}} \right) dy \\ & \leq C \int_{B_1(x) \setminus B_r(x)} \frac{1}{|y - x|^{N-2+2s}} \left[\left(1 - \frac{r^2}{|y - x|^2} \right)^{-s} - 1 \right] dy \\ & \leq C r^{2-2s} \int_1^{1/r} t^{1-2s} \left[\left(1 - \frac{1}{t^2} \right)^{-s} - 1 \right] dt \xrightarrow{r \downarrow 0} 0 \end{aligned}$$

as it can be checked for example with the L'Hôpital's rule. So any u which s -harmonic function in Ω satisfies $(-\Delta)^s u = 0$ in Ω . The converse follows from the following theorem.

THEOREM 2.1 | *Let $u : \mathbb{R}^N \rightarrow \mathbb{R}$ a measurable function such that for some open set $\Omega \subseteq \mathbb{R}^N$ is $u \in C^{2s+\varepsilon}(\Omega)$. Also, suppose that*

$$\int_{\mathbb{R}^N} \frac{|u(y)|}{1 + |y|^{N+2s}} dy < +\infty.$$

Then for any $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subseteq \Omega$, one has

$$u(x) = \int_{CB_r} \eta_r(y) u(x-y) dy + \gamma(N, s, r) (-\Delta)^s u(z), \quad \gamma(N, s, r) = \frac{\Gamma(N/2) r^{2s}}{2^{2s} \Gamma(\frac{N+2s}{2}) \Gamma(1+s)} \quad (2.1)$$

for some $z = z(x, s, r) \in \overline{B_r(x)}$.

Proof. Suppose, without loss of generality, that $x = 0$. Let $v = \Gamma_s - H$, where Γ_s is the fundamental solution¹ of the fractional Laplacian

$$\Gamma_s(x) = \frac{-C_{N,-s}}{|x|^{N-2s}}$$

and H solves in the pointwise sense

$$\begin{cases} (-\Delta)^s H = 0 & \text{in } B_r \\ H = \Gamma_s & \text{in } CB_r. \end{cases}$$

We claim that H satisfies equality

$$H(x) = H_1(x) := \int_{\mathbb{R}^N} \Gamma_s(x-y) \eta_r(y) dy = (\Gamma_s * \eta_r)(x).$$

Indeed

1. $(-\Delta)^s H_1 = 0$ in B_r because $H_1 = \Gamma_s * \eta_r$, $(-\Delta)^s \Gamma_s = \delta_0$ in the sense of distributions and $\eta_r = 0$ in B_r ,
2. since (cf. Appendix in [60, p. 399ss])

$$\int_{\mathbb{R}^N} \Gamma_s(x-y) \eta_r(y) dy = \int_{CB_r} \Gamma_s(x-y) \eta_r(y) dy = \Gamma_s(x), \quad |x| > r,$$

then $H_1 = \Gamma_s$ in CB_r , as desired.

Finally, as in 1., note that

$$(-\Delta)^s H(x) = \eta_r(x), \quad \text{when } |x| > r.$$

Since $u \in C^{2s+\varepsilon}(\Omega)$, $(-\Delta)^s u \in C(\Omega) \subseteq C(\overline{B_r})$, see [78, Proposition 2.4] or Lemma 2.3 below. Mollify u in order to obtain a sequence $\{u_j\}_j \subseteq C^\infty(\mathbb{R}^N)$. Hence,

$$\begin{aligned} \int_{B_r} v \cdot (-\Delta)^s u_j &= \int_{B_r} \Gamma_s \cdot (-\Delta)^s u_j - \int_{B_r} H \cdot (-\Delta)^s u_j = \\ &= \int_{\mathbb{R}^N} \Gamma_s \cdot (-\Delta)^s u_j - \int_{\mathbb{R}^N} H \cdot (-\Delta)^s u_j = u_j(0) - \int_{CB_r} u_j \eta_r, \end{aligned} \quad (2.2)$$

¹One possible construction and the explicit expression of the fundamental solution can be found in [18, paragraph 2.2].

where we have used the integration by parts formula (1.7) and the definition of Γ_s . On the one hand we have now that $(-\Delta)^s u_j \xrightarrow{j \uparrow + \infty} (-\Delta)^s u$ uniformly in B_r , since

$$\begin{aligned} & \sup_{x \in B_r} |(-\Delta)^s u_j(x) - (-\Delta)^s u(x)| = \\ &= \sup_{x \in B_r} \left| C_{N,s} PV \int_{\mathbb{R}^N} \frac{u_j(x) - u_j(y) - u(x) + u(y)}{|x - y|^{N+2s}} dy \right| \\ &\leq C_{N,s} \|u_j - u\|_{C^{2s+\varepsilon}(B_{r+\delta})} \sup_{x \in B_r} \int_{B_{r+\delta}} \frac{dy}{|x - y|^{N-\varepsilon}} + \\ &\quad + C_{N,s} \int_{\mathbb{R}^N \setminus B_{r+\delta}} \frac{\|u_j - u\|_{L^\infty(B_r)} + |u_j(y) - u(y)|}{(|y| - r)^{N+2s}} dy, \end{aligned}$$

while on the other hand

$$u_j(0) \xrightarrow{j \uparrow + \infty} u(0), \quad \int_{CB_r} u_j \eta_r \xrightarrow{j \uparrow + \infty} \int_{CB_r} u \eta_r$$

so that we can let $j \rightarrow +\infty$ in equality (2.2). Collecting the information so far, we have

$$u(0) = \int_{CB_r} u \eta_r + \int_{B_r} v (-\Delta)^s u = \int_{CB_r} u \eta_r + (-\Delta)^s u(z) \cdot \int_{B_r} v \quad (2.3)$$

for some $|z| \leq r$, by continuity of $(-\Delta)^s u$ in $\overline{B_r}$ and since $v > 0$ in $\overline{B_r}$. The constant $\gamma(N, s, r)$ appearing in the statement equals to $\int_{B_r} v$.

Let us compute $\gamma(N, s, r) = \int_{B_r} v$. If we consider the solution φ_δ to

$$\begin{cases} (-\Delta)^s \varphi_\delta = 1 & \text{in } B_{r+\delta} \\ \varphi_\delta = 0 & \text{in } CB_{r+\delta} \end{cases}$$

and we apply formula (2.3) to φ_δ in place of u to entail

$$\varphi_\delta(0) = \int_{CB_r} \varphi_\delta \eta_r + (-\Delta)^s \varphi_\delta(z) \cdot \int_{B_r} v = \int_{B_{r+\delta} \setminus B_r} \varphi_\delta \eta_r + \int_{B_r} v$$

The solution φ_δ is explicitly known (see [73, equation (1.4)] and references therein) and given by

$$\varphi_\delta(x) = \frac{\Gamma(N/2)}{2^{2s} \Gamma(\frac{N+2s}{2}) \Gamma(1+s)} ((r+\delta)^2 - |x|^2)^s.$$

Hence, by letting $\delta \downarrow 0$,

$$\gamma(N, s, r) = \int_{B_r} v = \lim_{\delta \downarrow 0} \varphi_\delta(0) = \frac{\Gamma(N/2)}{2^{2s} \Gamma(\frac{N+2s}{2}) \Gamma(1+s)} r^{2s}.$$

□

2.1.1 The Liouville theorem for s -harmonic functions

Following the proof of the classic Liouville Theorem due to Nelson [69], it is possible to prove the analogous result for the fractional Laplacian. In the following we denote by $\omega_{N-1} = \mathcal{H}^{N-1}(\partial B)$, the $(N-1)$ -dimensional Hausdorff measure of the unit sphere.

THEOREM 2.2 | *Let $u : \mathbb{R}^N \rightarrow \mathbb{R}$ be a function which is s -harmonic throughout \mathbb{R}^N . Then, if u is bounded in \mathbb{R}^N , it is constant.*

Proof. Take two arbitrary points $x_1, x_2 \in \mathbb{R}^N$. Both satisfy for all $r > 0$

$$u(x_1) = \int_{CB_r(x_1)} \eta_r(y - x_1) u(y) dy, \quad u(x_2) = \int_{CB_r(x_2)} \eta_r(y - x_2) u(y) dy.$$

Denote by $M := \sup_{\mathbb{R}^N} |u|$, which is finite by hypothesis:

$$\begin{aligned} |u(x_1) - u(x_2)| &= \left| \int_{CB_r(x_1)} \eta_r(y - x_1) u(y) dy - \int_{CB_r(x_2)} \eta_r(y - x_2) u(y) dy \right| \\ &\leq \int_{CB_r(x_1) \cap B_r(x_2)} \eta_r(y - x_1) M dy + \int_{CB_r(x_2) \cap B_r(x_1)} \eta_r(y - x_2) M dy \\ &\quad + \int_{CB_r(x_1) \cap CB_r(x_2)} |\eta_r(y - x_1) - \eta_r(y - x_2)| M dy. \end{aligned}$$

Define $\delta := |x_1 - x_2|$. The first addend (and similarly the second) vanish as $r \rightarrow +\infty$:

$$\begin{aligned} \int_{CB_r(x_1) \cap B_r(x_2)} \eta_r(y - x_1) dy &\leq \int_{B_{r+\delta} \setminus B_r} \frac{c(N, s) r^{2s}}{|y|^N (|y|^2 - r^2)^s} dy \\ &= \omega_{N-1} c(N, s) r^{2s} \int_r^{r+\delta} \frac{d\rho}{\rho (\rho^2 - r^2)^s} \\ &\leq \omega_{N-1} c(N, s) \frac{1}{r^{1-s}} \int_r^{r+\delta} \frac{d\rho}{(\rho - r)^s} \xrightarrow{r \uparrow +\infty} 0. \end{aligned}$$

The third one is more delicate:

$$\begin{aligned} &\int_{CB_r(x_1) \cap CB_r(x_2)} \left| \frac{r^{2s}}{|y - x_1|^N (|y - x_1|^2 - r^2)^s} - \frac{r^{2s}}{|y - x_2|^N (|y - x_2|^2 - r^2)^s} \right| dy \\ &= \int_{CB(x_1/r) \cap CB(x_2/r)} \left| \frac{1}{|y - \frac{x_1}{r}|^N (|y - \frac{x_1}{r}|^2 - 1)^s} - \frac{1}{|y - \frac{x_2}{r}|^N (|y - \frac{x_2}{r}|^2 - 1)^s} \right| dy \\ &= \int_{CB \cap CB(\frac{x_2 - x_1}{r})} \left| \frac{1}{|y|^N (|y|^2 - 1)^s} - \frac{1}{|y - \frac{x_2 - x_1}{r}|^N (|y - \frac{x_2 - x_1}{r}|^2 - 1)^s} \right| dy \\ &\quad [\text{setting } x_r = (x_2 - x_1)/r, |x_r| = \delta/r \rightarrow 0 \text{ as } r \rightarrow +\infty] \\ &= \int_{CB \cap CB(x_r)} \left| \frac{1}{|y|^N (|y|^2 - 1)^s} - \frac{1}{|y - x_r|^N (|y - x_r|^2 - 1)^s} \right| dy \\ &\quad [\text{taking wlog } x_r = \frac{\delta e_1}{r} \text{ and defining } H_r = \{x \in \mathbb{R}^N : x_1 > \frac{\delta}{2r}, |x - x_r| > 1\}] \\ &= 2 \int_{H_r} \left[\frac{1}{|y - x_r|^N (|y - x_r|^2 - 1)^s} - \frac{1}{|y|^N (|y|^2 - 1)^s} \right] dy \\ &\leq 2 \omega_{N-1} \int_1^{+\infty} \left[\frac{1}{\rho (\rho^2 - 1)^s} - \frac{\rho^{N-1}}{(\rho + \frac{\delta}{r})^N ((\rho + \frac{\delta}{r})^2 - 1)^s} \right] d\rho \\ &\leq 2 \omega_{N-1} \int_1^{+\infty} \frac{1}{\rho (1 + \frac{\delta}{r\rho})^N} \left[\frac{1}{(\rho^2 - 1)^s} - \frac{\rho^{N-1}}{((\rho + \frac{\delta}{r})^2 - 1)^s} \right] d\rho \xrightarrow{r \uparrow +\infty} 0 \end{aligned}$$

thanks to the Monotone Convergence Theorem. □

2.1.2 Asymptotics as $s \uparrow 1$

First of all, the proof of Theorem 2.1 implies

$$\lim_{s \uparrow 1} \gamma(N, s, r) = \gamma(N, 1, r) = \frac{\Gamma(N/2)}{4 \Gamma(\frac{N+2}{2}) \Gamma(2)} r^2 = \frac{r^2}{4} \cdot \frac{\Gamma(N/2)}{\frac{N}{2} \Gamma(N/2)} = \frac{r^2}{2N}.$$

Also

$$1 = \int_{CB_r} \eta_r(y) dy = \int_{CB_r} \frac{c(N, s) r^{2s}}{|y|^N (|y|^2 - r^2)^s} dy \quad \text{where } c(N, s) = \frac{2 \sin(\pi s)}{\pi \omega_{N-1}}.$$

Then

$$\begin{aligned} 1 &= \frac{2 \sin(\pi s)}{\pi \omega_{N-1}} \int_{CB_r} \frac{r^{2s}}{|y|^N (|y|^2 - r^2)^s} dy = \frac{2 \sin(\pi s)}{\pi \omega_{N-1}} \int_{CB} \frac{1}{|y|^N (|y|^2 - 1)^s} dy = \\ &= \frac{2 \sin(\pi s)}{\pi \omega_{N-1}} \int_{\partial B} \left[\int_1^{+\infty} \frac{d\rho}{\rho (\rho^2 - 1)^s} \right] d\mathcal{H}^{N-1}(\theta) = \frac{2 \sin(\pi s)}{\pi} \int_1^{+\infty} \frac{d\rho}{\rho (\rho^2 - 1)^s}. \end{aligned}$$

With similar computation

$$\int_{CB_r} \eta_r(y) u(x - y) dy = \frac{2 \sin(\pi s)}{\pi} \int_{\partial B} \left[\int_1^{+\infty} \frac{u(x - r\rho\theta)}{\rho (\rho^2 - 1)^s} d\rho \right] d\mathcal{H}^{N-1}(\theta).$$

Fix a $\theta \in \partial B$ and consider the difference

$$\left| \frac{2 \sin(\pi s)}{\pi} \int_1^{+\infty} \frac{u(x - r\rho\theta)}{\rho (\rho^2 - 1)^s} d\rho - u(x - r\theta) \right| \leq \frac{2 \sin(\pi s)}{\pi} \int_1^{+\infty} \frac{|u(x - r\rho\theta) - u(x - r\theta)|}{\rho (\rho^2 - 1)^s} d\rho :$$

since we are handling C^2 functions we can push a bit further the estimate to deduce

$$\begin{aligned} \left| \frac{2 \sin(\pi s)}{\pi} \int_1^{+\infty} \frac{u(x - r\rho\theta)}{\rho (\rho^2 - 1)^s} d\rho - u(x - r\theta) \right| &\leq \\ &\leq \frac{2 \sin(\pi s)}{\pi} \int_1^{1+\delta} \frac{C (\rho - 1)^{1-s}}{\rho (\rho + 1)^s} d\rho + \frac{2 \sin(\pi s)}{\pi} \int_{1+\delta}^{+\infty} \frac{|u(x - r\rho\theta) - u(x - r\theta)|}{\rho (\rho^2 - 1)^s} d\rho \xrightarrow{s \uparrow 1} 0 \end{aligned}$$

therefore

$$\int_{CB_r} \eta_r(y) u(x - y) dy \xrightarrow{s \uparrow 1} \frac{1}{\omega_{N-1} r^{N-1}} \int_{\partial B_r} u(x - y) d\mathcal{H}^{N-1}(y).$$

The choice of the point $z \in \overline{B_r}$ in (2.1) depends on the value of s , but since these are all points belonging to a compact set, we can build $\{s_k\}_{k \in \mathbb{N}} \subseteq (0, 1)$, $s_k \rightarrow 1$ as $k \rightarrow +\infty$, such that $z(s_k) \rightarrow z_0 \in \overline{B_r}$. Since it is known (see [37, Proposition 4.4]) that

$$\lim_{s \uparrow 1} (-\Delta)^s u = -\Delta u,$$

then

$$|(-\Delta)^s u(z(s_k)) + \Delta u(z_0)| \leq |(-\Delta)^s u(z(s_k)) - (-\Delta)^s u(z_0)| + |(-\Delta)^s u(z_0) + \Delta u(z_0)| \xrightarrow{k \rightarrow +\infty} 0.$$

Finally we have proven that

$$\begin{aligned} u(x) &= \int_{CB_r} \eta_r(y) u(x - y) dy + \gamma(N, s, r) (-\Delta)^s u(z(s_k)) \\ &\xrightarrow{k \rightarrow +\infty} \frac{1}{\omega_{N-1} r^{N-1}} \int_{\partial B_r} u(x - y) d\mathcal{H}^{N-1}(y) - \frac{r^2}{2n} \Delta u(z_0) \end{aligned}$$

which is a known formula for C^2 functions, see e.g. [41, Proposition A.1.2].

2.2 EXISTENCE AND UNIQUENESS

Assume $\Omega \subseteq \mathbb{R}^N$ is open and bounded, with $C^{1,1}$ boundary.

2.2.1 Proof of Proposition 1.2

Assume first that $u \in \mathcal{S}$ and $v = 0$ in $\mathbb{R}^N \setminus \Omega$, $v \in C^{2s+\varepsilon}(\Omega) \cap C(\overline{\Omega})$ and $(-\Delta)^s v \in L^1(\mathbb{R}^N)$; then we can regularize v , via the convolution with a mollifier $\{\alpha_k(x) = k^N \alpha(kx) : \alpha(x) = 0 \text{ for } |x| \geq 1\}_{k \in \mathbb{N}}$ in order to obtain a sequence $\{v_k := \alpha_k * v\}_{k \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^N) \subseteq \mathcal{S}$ converging uniformly to v in \mathbb{R}^N . Also,

$$(-\Delta)^s v_k = v * (-\Delta)^s \alpha_k$$

indeed

$$\begin{aligned} (-\Delta)^s v_k(x) &= C_{N,s} PV \int_{\mathbb{R}^N} \frac{v_k(x) - v_k(y)}{|x - y|^{N+2s}} dy \\ &= C_{N,s} \lim_{\varepsilon \downarrow 0} \int_{CB_\varepsilon(x)} \frac{\int_{\mathbb{R}^N} v(z) [\alpha_k(x - z) - \alpha_k(y - z)] dz}{|x - y|^{N+2s}} dy \\ &= C_{N,s} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} v(z) \int_{CB_\varepsilon(x)} \frac{\alpha_k(x - z) - \alpha_k(y - z)}{|x - y|^{N+2s}} dy dz \\ &= C_{N,s} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^N} v(z) \int_{CB_\varepsilon(x-z)} \frac{\alpha_k(x - z) - \alpha_k(y)}{|x - z - y|^{N+2s}} dy dz \end{aligned}$$

where we have

$$C_{N,s} \int_{CB_\varepsilon(x)} \frac{\alpha_k(x) - \alpha_k(y)}{|x - y|^{N+2s}} dy \xrightarrow{\varepsilon \downarrow 0} (-\Delta)^s \alpha_k(x), \quad \text{uniformly in } \mathbb{R}^N$$

since, see [37, Lemma 3.2],

$$\begin{aligned} &\left| (-\Delta)^s \alpha_k(x) - C_{N,s} \int_{CB_\varepsilon(x)} \frac{\alpha_k(x) - \alpha_k(y)}{|x - y|^{N+2s}} dy \right| = \\ &= \left| C_{N,s} PV \int_{B_\varepsilon(x)} \frac{\alpha_k(x) - \alpha_k(y)}{|x - y|^{N+2s}} dy \right| \end{aligned} \tag{2.4}$$

$$\begin{aligned} &= \left| \frac{C_{N,s}}{2} \int_{B_\varepsilon} \frac{\alpha_k(x+y) + \alpha_k(x-y) - 2\alpha_k(x)}{|y|^{N+2s}} dy \right| \\ &\leq \frac{C_{N,s} \|\alpha_k\|_{C^2(B_\varepsilon(x))}}{2} \int_{B_\varepsilon} \frac{dy}{|y|^{N+2s-2}} \\ &\leq \frac{C_{N,s} \|\alpha_k\|_{C^2(\mathbb{R}^N)}}{2} \int_{B_\varepsilon} \frac{dy}{|y|^{N+2s-2}} \xrightarrow{\varepsilon \downarrow 0} 0. \end{aligned} \tag{2.5}$$

Following the very same proof up to (2.5), since $v \in C^{2s+\varepsilon}(\Omega)$, it is also possible to prove

$$(-\Delta)^s v_k(x) = (\alpha_k * (-\Delta)^s v)(x), \quad \text{for } \delta(x) > \frac{1}{k}.$$

Since $(-\Delta)^s v \in C(\mathbb{R}^N \setminus \Omega)$, see [78, Proposition 2.4], we infer that

$$(-\Delta)^s v_k(x) \xrightarrow{k \uparrow +\infty} (-\Delta)^s v, \quad \text{for every } x \in \mathbb{R}^N \setminus \partial\Omega.$$

We give now a pointwise estimate on $(-\Delta)^s \alpha_k(x)$, $x \in B_{1/k}$. Since $\alpha_k \in C_c^\infty(\mathbb{R}^N)$, we can write (see [78, Paragraph 2.1])

$$\begin{aligned} (-\Delta)^s \alpha_k(x) &= \left[(-\Delta)^{s-1} \circ (-\Delta) \right] \alpha_k(x) = (-\Delta)^{s-1} [-k^{N+2} \Delta \alpha(kx)] = \\ &= C_{N,-s+1} \int_{\mathbb{R}^N} \frac{k^{N+2} \Delta \alpha(ky)}{|x - y|^{N+2s-2}} dy. \end{aligned}$$

With a change of variable we entail

$$(-\Delta)^s \alpha_k(x) = C_{N,-s+1} k^{N+2s} \int_B \frac{\Delta \alpha(y)}{|kx - y|^{N+2s-2}} dy$$

and therefore

$$|(-\Delta)^s \alpha_k(x)| \leq \frac{\omega_{N-1}}{2(1-s)} k^{N+2s} \|\Delta \alpha\|_{L^\infty(\mathbb{R}^N)}. \quad (2.6)$$

Indeed,

$$\int_B \frac{dy}{|z - y|^{N+2s-2}}$$

is a bounded function of z , having in $z = 0$ its maximum $\frac{\omega_{N-1}}{2(1-s)}$. Indeed,

$$\begin{aligned} \int_B \frac{dy}{|z - y|^{N+2s-2}} &= \int_{B \cap B(z)} \frac{dy}{|z - y|^{N+2s-2}} + \int_{B \setminus B(z)} \frac{dy}{|z - y|^{N+2s-2}} \\ &= \int_{B \cap B(z)} \frac{dy}{|y|^{N+2s-2}} + \int_{B \setminus B(z)} \frac{dy}{|z - y|^{N+2s-2}} \\ &\leq \int_{B \cap B(z)} \frac{dy}{|y|^{N+2s-2}} + \int_{B \setminus B(z)} \frac{dy}{|y|^{N+2s-2}} = \int_B \frac{dy}{|y|^{N+2s-2}}. \end{aligned}$$

Using (2.6), the L^1 -norm of $(-\Delta)^s v_k$ can be estimated by

$$\begin{aligned} \int_{\mathbb{R}^N} |(-\Delta)^s v_k(x)| dx &= \int_{\{\delta(x) < 1/k\}} |(v * (-\Delta)^s \alpha_k)(x)| dx + \int_{\{\delta(x) \geq 1/k\}} |(\alpha_k * (-\Delta)^s v)(x)| dx \\ &\leq \int_{\{\delta(x) < 1/k\}} \int_{B_{1/k}(x)} |v(y)(-\Delta)^s \alpha_k(x - y)| dy dx + \int_{\{\delta(x) \geq 1/k\}} \int_{B_{1/k}(x)} |\alpha_k(y)(-\Delta)^s v(x - y)| dy dx \\ &\leq \int_{\{\delta(x) < 1/k\}} \int_{B_{1/k}(x)} C \delta(y)^s k^{N+2s} \|\Delta \alpha\|_{L^\infty(\mathbb{R}^N)} dy dx + \int_{B_{1/k}} \alpha_k(y) \int_{\mathbb{R}^N} |(-\Delta)^s v(x - y)| dx dy \\ &\leq \frac{C \|\Delta \alpha\|_{L^\infty(\mathbb{R}^N)}}{k^{1-s}} + \|(-\Delta)^s v\|_{L^1(\mathbb{R}^N)}, \end{aligned}$$

so that, by the Fatou's Lemma we have

$$\int_{\mathbb{R}^N} |(-\Delta)^s v| \leq \liminf_{k \uparrow +\infty} \int_{\mathbb{R}^N} |(-\Delta)^s v_k| \leq \limsup_{k \uparrow +\infty} \int_{\mathbb{R}^N} |(-\Delta)^s v_k| \leq \int_{\mathbb{R}^N} |(-\Delta)^s v|$$

which means

$$\|(-\Delta)^s v_k\|_{L^1(\mathbb{R}^N)} \xrightarrow{k \uparrow +\infty} \|(-\Delta)^s v\|_{L^1(\mathbb{R}^N)}.$$

Apply the Fatou's lemma to $|(-\Delta)^s v_k| + |(-\Delta)^s v| - |(-\Delta)^s v_k - (-\Delta)^s v| \geq 0$ to deduce

$$\begin{aligned} 2 \int_{\mathbb{R}^N} |(-\Delta)^s v| &\leq \liminf_{k \uparrow +\infty} \int_{\mathbb{R}^N} (|(-\Delta)^s v_k| + |(-\Delta)^s v| - |(-\Delta)^s v_k - (-\Delta)^s v|) = \\ &= 2 \int_{\mathbb{R}^N} |(-\Delta)^s v| - \limsup_{k \uparrow +\infty} \int_{\mathbb{R}^N} |(-\Delta)^s v_k - (-\Delta)^s v| \end{aligned}$$

and conclude

$$\|(-\Delta)^s v_k - (-\Delta)^s v\|_{L^1(\mathbb{R}^N)} \xrightarrow{k \uparrow +\infty} 0$$

and, for any $u \in \mathcal{S}$,

$$\int_{\mathbb{R}^N} u(-\Delta)^s v_k \xrightarrow{k \uparrow +\infty} \int_{\mathbb{R}^N} u(-\Delta)^s v.$$

Note now that

$$\int_{\Omega_k} v_k(-\Delta)^s u = \int_{\mathbb{R}^N} v_k(-\Delta)^s u \xrightarrow{k \uparrow 0} \int_{\mathbb{R}^N} v(-\Delta)^s u = \int_{\Omega} v(-\Delta)^s u$$

since $\|v_k - v\|_{L^\infty(\mathbb{R}^N)} \xrightarrow{k \uparrow 0} 0$, and this concludes the proof.

2.2.2 Preliminaries on fractional Green functions, Poisson kernels and Martin kernels

Consider the function $G_\Omega : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ built as the family of solutions to the problems

$$\text{for any } x \in \Omega \quad \begin{cases} (-\Delta)^s G_\Omega(x, \cdot) = \delta_x & \text{in } \Omega, \\ G_\Omega(x, y) = 0 & \text{in } \mathcal{C}\Omega. \end{cases}$$

This function can be written as the sum

$$G_\Omega(x, y) = \Gamma_s(y - x) - H(x, y),$$

where Γ_s is the fundamental solution to the fractional Laplacian, and

$$\text{for any } x \in \Omega \quad \begin{cases} (-\Delta)^s H(x, \cdot) = 0 & \text{in } \Omega, \\ H(x, y) = \Gamma_s(y - x) & \text{in } \mathcal{C}\Omega. \end{cases}$$

LEMMA 2.3 | Fix $x \in \Omega$. Then $H(x, \cdot) \in C^{2s+\varepsilon}(\Omega) \cap C(\overline{\Omega})$.

Proof. Take $r = r(x) > 0$ such that $\overline{B_r(x)} \subseteq \Omega$ and κ a cutoff function

$$0 \leq \kappa \leq 1 \text{ in } \mathbb{R}^N, \quad \kappa \in C^\infty(\mathbb{R}^N), \quad \kappa = 1 \text{ in } \overline{\mathcal{C}\Omega}, \quad \kappa = 0 \text{ in } \overline{B_r(x)}$$

and define $\gamma_s(y) := \kappa(y)\Gamma_s(y - x)$: $\gamma_s \in C^\infty(\mathbb{R}^N)$, $\gamma_s = \Gamma_s$ in $\mathcal{C}\Omega$, $\gamma_s = 0$ in $B_r(x)$. Then establish the equivalent problem

$$\forall x \in \Omega \quad \begin{cases} (-\Delta)^s H(x, \cdot) = 0 & \text{in } \Omega \\ H(x, y) = \gamma_s(y - x) & \text{in } \mathcal{C}\Omega \end{cases}$$

and by setting $h(x, y) := H(x, y) - \gamma_s(x, y)$ we obtain

$$\forall x \in \Omega \quad \begin{cases} (-\Delta)^s h(x, \cdot) = -(-\Delta)^s \gamma_s & \text{in } \Omega, \\ h(x, y) = 0 & \text{in } \mathcal{C}\Omega. \end{cases}$$

Note that $(-\Delta)^s \gamma_s \in L^\infty(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ (see [78, Proposition 2.7]): from this we deduce that $h(x, \cdot) \in C^{2s+\varepsilon}(\Omega) \cap C^s(\mathbb{R}^N)$ by [78, Proposition 2.8] and [73, Proposition 1.1] respectively and then also $H(x, \cdot) \in C^{2s+\varepsilon}(\Omega) \cap C^s(\mathbb{R}^N)$. \square

LEMMA 2.4 | The function $G_\Omega(x, y)$ we have just obtained satisfies the following properties:

- i) G_Ω is continuous in $\Omega \times \Omega$ except on the diagonal²,
- ii) $(-\Delta)^s G_\Omega(x, \cdot) \in L^1(\mathcal{C}\Omega)$ for any $x \in \Omega$,
- iii) for any $u \in C^{2s+\varepsilon}(\Omega) \cap L^\infty(\mathbb{R}^N)$ and $x \in \Omega$

$$u(x) = \int_{\Omega} (-\Delta)^s u(y) G_\Omega(x, y) dy - \int_{\mathcal{C}\Omega} u(y) (-\Delta)^s G_\Omega(x, y) dy, \quad (2.7)$$

and this formula is a Green's representation formula (G_Ω is the Green function while its fractional Laplacian is the Poisson kernel),

- iv) $(-\Delta)^s G_\Omega(x, y)$, $x \in \Omega$, $y \in \mathcal{C}\Omega$ is given by the formula

$$(-\Delta)^s G_\Omega(x, y) = -C_{N,s} \int_{\Omega} \frac{G_\Omega(x, z)}{|z - y|^{N+2s}} dz, \quad (2.8)$$

- v) $G_\Omega(x, y) \geq 0$ for a.e. $(x, y) \in \Omega \times \Omega$, $x \neq y$, and $(-\Delta)^s G_\Omega(x, y) \leq 0$ for any $x \in \Omega$, $y \in \mathcal{C}\Omega$,

²where its singularity is inherited by the singularity in 0 of Γ_s

vi) it holds

$$\Gamma_s(y-x) = - \int_{\mathcal{C}\Omega} \Gamma_s(y-z) \cdot (-\Delta)^s G_\Omega(x, z) dz \quad \text{for } x \in \Omega, y \in \mathcal{C}\Omega, \quad (2.9)$$

$$G_\Omega(x, y) - \Gamma_s(y-x) = - \int_{\mathcal{C}\Omega} \Gamma_s(y-z) \cdot (-\Delta)^s G_\Omega(x, z) dz \quad \text{for } x \in \Omega, y \in \Omega. \quad (2.10)$$

Proof. We prove all conclusions step by step.

Proof of ii). First of all, we use the estimate

$$|h(x, y)| \leq C \|(-\Delta)^s \gamma_s(x, \cdot)\|_\infty \delta(y)^s,$$

where h solves

$$\forall x \in \Omega \quad \begin{cases} (-\Delta)^s h(x, \cdot) = -(-\Delta)^s \gamma_s & \text{in } \Omega \\ h(x, y) = 0 & \text{in } \mathcal{C}\Omega \end{cases}$$

and γ_s is a regularization of Γ_s as in Lemma 2.3; for the inequality we refer to [73, Proposition 1.1]. We deduce that, for y sufficiently close to $\partial\Omega$, it holds

$$|G_\Omega(x, y)| = |\Gamma_s(y-x) - H(x, y)| = |\gamma_s(y-x) - H(x, y)| = |h(x, y)| \leq C \|(-\Delta)^s \gamma_s(x, \cdot)\|_\infty \delta(y)^s.$$

Then, for $y \in \mathcal{C}\Omega$,

$$\begin{aligned} |(-\Delta)^s G_\Omega(x, y)| &= \left| C_{N,s} PV \int_{\mathbb{R}^N} \frac{G_\Omega(x, z) - G_\Omega(x, y)}{|y-z|^{N+2s}} dz \right| = \\ &= C_{N,s} \int_{\Omega} \frac{G_\Omega(x, z)}{|y-z|^{N+2s}} dz \leq C \int_{\Omega} \frac{\delta(z)^s}{|y-z|^{N+2s}} dz \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} \int_{\mathcal{C}\Omega} |(-\Delta)^s G_\Omega(x, y)| dy &\leq C \int_{\mathcal{C}\Omega} \int_{\Omega} \frac{\delta(z)^s}{|y-z|^{N+2s}} dz dy \leq \\ &\leq C \int_{\Omega} \delta(z)^s \int_{\delta(z)}^{+\infty} \rho^{-1-2s} d\rho dz \leq \frac{C}{2s} \int_{\Omega} \frac{dz}{\delta(z)^s} < +\infty. \end{aligned} \quad (2.12)$$

Proof of iii). The function $G_\Omega(x, y) = \Gamma_s(y-x) - H(x, y)$ can be integrated by parts against any $u \in C^{2s+\varepsilon}(\mathbb{R}^N \setminus \Omega) \cap L^\infty(\mathbb{R}^N)$, since, according to Lemma 2.3, $H(x, \cdot) \in C^{2s+\varepsilon}(\Omega) \cap C(\overline{\Omega})$ and so we can apply Proposition 1.2. Hence,

$$u(x) = \int_{\Omega} (-\Delta)^s u(y) G_\Omega(x, y) dy - \int_{\mathcal{C}\Omega} u(y) (-\Delta)^s G_\Omega(x, y) dy.$$

Proof of iv). We point how the computation of $(-\Delta)^s G_\Omega(x, y)$, $x \in \Omega, y \in \mathcal{C}\Omega$ reduces to a more readable formula:

$$-(-\Delta)^s G_\Omega(x, y) = C_{N,s} PV \int_{\mathbb{R}^N} \frac{G_\Omega(x, z) - G_\Omega(x, y)}{|z-y|^{N+2s}} dz = C_{N,s} \int_{\Omega} \frac{G_\Omega(x, z)}{|z-y|^{N+2s}} dz,$$

by simply recalling that $G_\Omega(x, z) = 0$, when $z \in \mathcal{C}\Omega$.

Proof of v). $G_\Omega(x, y) \geq 0$ for $(x, y) \in \Omega \times \Omega, x \neq y$, in view of Lemma 2.8 below applied to the function $H(x, \cdot)$. Also, from this and (2.8) we deduce that

$$-(-\Delta)^s G_\Omega(x, y) = C_{N,s} \int_{\Omega} \frac{G_\Omega(x, z)}{|z-y|^{N+2s}} dz \geq 0. \quad (2.13)$$

Proof of vi). It suffices to apply (2.7) to the solution $H(x, y)$ of

$$\begin{cases} (-\Delta)^s H(x, \cdot) = 0 & \text{in } \Omega \\ H(x, y) = \Gamma_s(y-x) & \text{in } \mathcal{C}\Omega \end{cases}$$

to infer

$$\begin{aligned}\Gamma_s(y-x) &= - \int_{\mathcal{C}\Omega} \Gamma_s(y-z) \cdot (-\Delta)^s G_\Omega(x, z) \, dz & \text{for } x \in \Omega \text{ and } y \in \mathcal{C}\Omega, \\ H(x, y) &= - \int_{\mathcal{C}\Omega} \Gamma_s(y-z) \cdot (-\Delta)^s G_\Omega(x, z) \, dz & \text{for } x \in \Omega \text{ and } y \in \Omega.\end{aligned}$$

□

LEMMA 2.5 | For any $x \in \Omega$, $\theta \in \partial\Omega$ the function

$$M_\Omega(x, \theta) = \lim_{\substack{y \in \Omega \\ y \rightarrow \theta}} \frac{G_\Omega(x, y)}{\delta(y)^s}$$

is well-defined in $\Omega \times \partial\Omega$ and for any $h \in C(\partial\Omega)$, $\psi \in C_c^\infty(\Omega)$ one has

$$\int_\Omega \left(\int_{\partial\Omega} M_\Omega(x, \theta) h(\theta) \, d\mathcal{H}(\theta) \right) \psi(x) \, dx = \int_{\partial\Omega} h(\theta) D_s \phi(\theta) \, d\mathcal{H}(\theta),$$

where

$$D_s \phi(\theta) = \lim_{\substack{y \in \Omega \\ y \rightarrow \theta}} \frac{\phi(y)}{\delta(y)^s} \quad \text{and} \quad \phi(y) = \int_\Omega G_\Omega(x, y) \psi(x) \, dx.$$

Proof. For a small parameter $\varepsilon > 0$ define $\Omega_\varepsilon = \{x \in \overline{\Omega} : 0 \leq \delta(x) < \varepsilon\}$: associate to any $x \in \Omega$ a couple (ρ, θ) where $\rho = \delta(x)$ and $\theta \in \partial\Omega$ satisfies $|x - \theta| = \rho$: such a θ is uniquely determined for small ε since $\partial\Omega \in C^{1,1}$. Take also $\varphi \in C^\infty(\mathbb{R})$, with $\varphi(0) = 1$ and supported in $[-1, 1]$. With a slight abuse of notation define

$$f_\varepsilon(x) = f_\varepsilon(\rho, \theta) = h(\theta) \frac{\varphi(\rho/\varepsilon) \rho^{-s}}{K_\varepsilon}, \quad K_\varepsilon = \frac{1}{1+s} \int_0^\varepsilon \varphi(r/\varepsilon) \, dr. \quad (2.14)$$

Consider the functions

$$u_\varepsilon(x) = \int_\Omega f_\varepsilon(y) G_\Omega(x, y) \, dy,$$

and any function $\psi \in C_c^\infty(\Omega)$:

$$\begin{aligned}\int_\Omega u_\varepsilon \psi &= \int_\Omega f_\varepsilon(y) \left[\int_\Omega \psi(x) G_\Omega(x, y) \, dx \right] dy = \int_{\Omega_\varepsilon} h(\theta(y)) \frac{\varphi(\delta(y)/\varepsilon) \delta(y)^{-s}}{K_\varepsilon} \left[\int_\Omega \psi(x) G_\Omega(x, y) \, dx \right] dy \\ &= \int_{\Omega_\varepsilon} h(\theta(y)) \frac{\varphi(\rho(y)/\varepsilon)}{K_\varepsilon} \left[\int_\Omega \psi(x) \frac{G_\Omega(x, y)}{\delta(y)^s} \, dx \right] dy.\end{aligned}$$

Note that the function $\psi_1(y) = \int_\Omega \psi(x) G_\Omega(x, y) \delta(y)^{-s} \, dx$ is continuous in $\overline{\Omega_\varepsilon}$, as a consequence of the boundary behaviour of G_Ω , see [27, equation (2.13)]. Then

$$\begin{aligned}\int_{\Omega_\varepsilon} h(\theta(y)) \frac{\varphi(\rho(y)/\varepsilon)}{K_\varepsilon} \psi_1(y) \, dy &= \frac{1}{(1+s) K_\varepsilon} \int_0^\varepsilon \varphi(r/\varepsilon) \left(\int_{\partial\Omega} h(\theta) \psi_1(r, \theta) \, d\mathcal{H}(\theta) \right) dr \\ &\xrightarrow{\varepsilon \downarrow 0} \int_{\partial\Omega} h(\theta) \psi_1(0, \theta) \, d\mathcal{H}(\theta) = \int_{\partial\Omega} h(\theta) \int_\Omega \psi(x) \lim_{y \rightarrow \theta} \frac{G_\Omega(x, y)}{\delta(y)^s} \, dx \, d\mathcal{H}(\theta).\end{aligned}$$

Hence

$$\int_\Omega u_\varepsilon \psi \xrightarrow{\varepsilon \downarrow 0} \int_\Omega \left(\int_{\partial\Omega} M_\Omega(x, \theta) h(\theta) \, d\mathcal{H}(\theta) \right) \psi(x) \, dx \quad (2.15)$$

where

$$M_\Omega(x, \theta) = \lim_{\substack{y \in \Omega \\ y \rightarrow \theta}} \frac{G_\Omega(x, y)}{\delta(y)^s}.$$

Note that, always in view of the boundary estimates on G_Ω , the function $M_\Omega(x, \cdot) \in L^\infty(\partial\Omega)$ for any fixed $x \in \Omega$. In addition,

$$\begin{aligned} \int_\Omega u_\varepsilon \psi &= \int_{\Omega_\varepsilon} h(\theta(y)) \frac{\varphi(\rho(y)/\varepsilon)}{K_\varepsilon} \left[\int_\Omega \psi(x) G_\Omega(x, y) dx \right] dy = \\ &= \int_{\Omega_\varepsilon} h(\theta(y)) \frac{\varphi(\rho(y)/\varepsilon) \delta(y)^s}{K_\varepsilon \delta(y)^s} \phi(y) dy \xrightarrow{\varepsilon \downarrow 0} \int_{\partial\Omega} h(\theta) D_s \phi(\theta) d\mathcal{H}(\theta). \end{aligned} \quad (2.16)$$

So the limits (2.15) and (2.16) must coincide. \square

REMARK 2.6 | The function $M_\Omega(x, \theta)$ we have just introduced is closely related to the *Martin kernel based at $x_0 \in \Omega$*

$$\widetilde{M}_\Omega(x, \theta) = \lim_{\substack{y \in \Omega \\ y \rightarrow \theta}} \frac{G_\Omega(x, y)}{G_\Omega(x_0, y)}.$$

For this reason we borrow the usual notation of the Martin kernel.

LEMMA 2.7 | For any $h \in C(\partial\Omega)$ define

$$u(x) = \int_{\partial\Omega} M_\Omega(x, \theta) h(\theta) d\mathcal{H}(\theta), \quad x \in \Omega.$$

Then for any $\theta^* \in \partial\Omega$

$$Eu(\theta^*) := \lim_{\substack{x \in \Omega \\ x \rightarrow \theta^*}} \frac{u(x)}{\int_{\partial\Omega} M_\Omega(x, \theta) d\mathcal{H}(\theta)} = h(\theta^*).$$

Proof. Denote by

$$L(x) = \delta(x)^{1-s} \int_{\partial\Omega} M_\Omega(x, \theta) d\mathcal{H}(\theta), \quad x \in \Omega$$

for which we have

$$\int_{\partial\Omega} \frac{\delta(x)}{c|x-\theta|^N} d\mathcal{H}(\theta) \leq L(x) \leq \int_{\partial\Omega} \frac{c\delta(x)}{|x-\theta|^N} d\mathcal{H}(\theta),$$

and so L is a bounded quantity. Indeed, referring to estimates on the Green function in [27, equation (2.13)], we have inequalities

$$\frac{\delta(x)^s}{c|x-\theta|^N} \leq M_\Omega(x, \theta) \leq \frac{c\delta(x)^s}{|x-\theta|^N}, \quad x \in \Omega, \theta \in \partial\Omega. \quad (2.17)$$

Thus,

$$\begin{aligned} \left| \frac{\delta(x)^{1-s} u(x)}{L(x)} - h(\theta^*) \right| &= \left| \frac{\delta(x)^{1-s}}{L(x)} \int_{\partial\Omega} M_\Omega(x, \theta) h(\theta) d\mathcal{H}(\theta) - h(\theta^*) \frac{\delta(x)^{1-s}}{L(x)} \int_{\partial\Omega} M_\Omega(x, \theta) d\mathcal{H}(\theta) \right| \leq \\ &\leq \frac{\delta(x)^{1-s}}{L(x)} \int_{\partial\Omega} M_\Omega(x, \theta) |h(\theta) - h(\theta^*)| d\mathcal{H}(\theta) \leq C\delta(x) \int_{\partial\Omega} \frac{|h(\theta) - h(\theta^*)|}{|x-\theta|^N} d\mathcal{H}(\theta). \end{aligned}$$

Call $\theta_x \in \partial\Omega$ the point satisfying $\delta(x) = |x - \theta_x|$. Describe $\partial\Omega$ as a graph in a neighbourhood of 0, i.e. $\Gamma \subseteq \partial\Omega$ open, and

$$\begin{aligned} \Gamma \ni \theta &= (\theta', \phi(\theta')) \text{ for some } \phi : B'_r \subseteq \mathbb{R}^{N-1} \rightarrow \mathbb{R}, \quad (0, \phi(0)) = \theta_x, \\ \phi &\in C^{1,1}(B_r(0)), \quad \nabla \phi(0) = 0. \end{aligned}$$

It holds $\theta^* \in \Gamma$ for x close enough to θ^* . Let us now write,

$$\int_\Gamma \frac{|h(\theta) - h(\theta^*)|}{|x-\theta|^N} d\mathcal{H}(\theta) \leq \int_\Gamma \frac{|h(\theta) - h(\theta^*)|}{[|x-\theta_x|^2 + |\theta_x - \theta|^2 - 2\langle x-\theta_x, \theta_x - \theta \rangle]^{N/2}} d\mathcal{H}(\theta)$$

where

$$2\langle x - \theta_x, \theta_x - \theta \rangle = 2|x - \theta_x| \cdot |\theta_x - \theta| \left\langle \frac{x - \theta_x}{|x - \theta_x|}, \frac{\theta_x - \theta}{|\theta_x - \theta|} \right\rangle \leq 2\mu|x - \theta_x| \cdot |\theta_x - \theta|, \quad \theta \in \Gamma$$

for some $\mu < 1$, since $x - \theta_x$ is normal to $\partial\Omega$, while $\theta_x - \theta$ is “almost tangent” to $\partial\Omega$. Then

$$\int_{\Gamma} \frac{|h(\theta) - h(\theta^*)|}{|x - \theta|^N} d\mathcal{H}(\theta) \leq \frac{1}{(1 - \mu)^{N/2}} \int_{\Gamma} \frac{|h(\theta) - h(\theta^*)|}{[|x - \theta_x|^2 + |\theta_x - \theta|^2]^{N/2}} d\mathcal{H}(\theta)$$

Suppose without loss of generality that $\theta_x = 0$ and denote by ω the modulus of continuity of h :

$$\begin{aligned} \int_{\Gamma} \frac{|h(\theta) - h(\theta^*)|}{|x - \theta|^N} d\mathcal{H}(\theta) &\leq \frac{\sup_{\Gamma} \omega(|\theta - \theta^*|)}{(1 - \mu)^{N/2}} \int_{B_r'} \frac{d\theta'}{[|x|^2 + |\theta'|^2 + |\phi(\theta')|^2]^{N/2}} \leq \\ &\leq \frac{\sup_{\Gamma} \omega(|\theta - \theta^*|)}{(1 - \mu)^{N/2}} \int_0^r \frac{\rho^{N-2} d\rho}{[|x|^2 + \rho^2]^{N/2}} \leq \\ &\leq \frac{\sup_{\Gamma} \omega(|\theta - \theta^*|)}{(1 - \mu)^{N/2}} \int_0^r \frac{d\rho}{|x|^2 + \rho^2} \leq \frac{\sup_{\Gamma} \omega(|\theta - \theta^*|)}{|x| (1 - \mu)^{N/2}} \end{aligned}$$

and therefore

$$\begin{aligned} \delta(x) \int_{\partial\Omega} \frac{|h(\theta) - h(\theta^*)|}{|x - \theta|^N} d\mathcal{H}(\theta) &= \delta(x) \int_{\Gamma} \frac{|h(\theta) - h(\theta^*)|}{|x - \theta|^N} d\mathcal{H}(\theta) + \delta(x) \int_{\partial\Omega \setminus \Gamma} \frac{|h(\theta) - h(\theta^*)|}{|x - \theta|^N} d\mathcal{H}(\theta) \leq \\ &\leq \frac{\sup_{\Gamma} \omega(|\theta - \theta^*|)}{(1 - \mu)^{N/2}} + o(\delta(x)). \end{aligned}$$

Now, since $\sup_{\Gamma} \omega(|\theta - \theta^*|) \leq \omega(\text{diam}\Gamma)$, where $\text{diam}\Gamma = \sup_{\theta, \theta' \in \Gamma} |\theta - \theta'|$, and Γ is arbitrary, we deduce

$$\delta(x) \int_{\partial\Omega} \frac{|h(\theta) - h(\theta^*)|}{|x - \theta|^N} d\mathcal{H}(\theta) \xrightarrow{x \rightarrow \theta^*} 0.$$

□

2.2.3 Linear theory for smooth data: proof of Theorem 1.4

We start by stating

LEMMA 2.8: MAXIMUM PRINCIPLE | Let $\mathcal{O} \subset \mathbb{R}^N$ be an open set such that $\overline{\Omega} \subset \mathcal{O}$. Let $u : \mathbb{R}^N \rightarrow \mathbb{R}$ be a measurable function in $C^{2s+\varepsilon}(\Omega) \cap C(\mathcal{O})$, and

$$\int_{\mathbb{R}^N} \frac{|u(y)|}{1 + |y|^{N+2s}} dy < +\infty, \quad \text{and} \quad \begin{cases} (-\Delta)^s u \leq 0 & \text{in } \Omega \\ u \leq 0 & \text{in } \mathcal{C}\Omega. \end{cases}$$

Then $u \leq 0$ in Ω .

Proof. Call $\Omega^+ = \{x \in \mathbb{R}^N : u > 0\}$; Ω^+ is an open set contained in Ω . Assume by contradiction that $\Omega^+ \neq \emptyset$. By continuity of u in $\overline{\Omega^+} \subset \mathcal{O}$, there exists $x_0 \in \overline{\Omega^+}$ such that $u(x_0) = \max\{u(x) : x \in \overline{\Omega^+}\}$; moreover, $x_0 \in \Omega^+$, since $u \leq 0$ on $\partial\Omega^+$. This point x_0 will be also a global maximum for u since outside Ω^+ the function u is nonpositive. Thus

$$(-\Delta)^s u(x_0) = C_{N,s} PV \int_{\mathbb{R}^N} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2s}} dy > 0$$

contradicting our hypotheses. Therefore Ω^+ is empty. □

By splitting g into its positive and negative part, it suffices to prove Theorem 1.4 in the case where $g \geq 0$. So, from now on we will deal with nonnegative boundary data $g : \mathcal{C}\Omega \rightarrow [0, +\infty)$ which are measurable functions with

$$0 \leq - \int_{\mathcal{C}\Omega} g(y) \cdot (-\Delta)^s G_\Omega(x, y) dy < +\infty \quad x \in \Omega. \quad (2.18)$$

Note that, in view of equation (2.11), for $x \in \Omega$, $y \in \mathcal{C}\Omega$

$$0 \leq -(-\Delta)^s G_\Omega(x, y) \leq C \int_{\mathcal{C}\Omega} \frac{\delta(y)^s}{|y - z|^{N+2s}} dz \leq C \int_{\delta(y)}^{+\infty} \frac{d\rho}{\rho^{1+s}} = \frac{C}{s} \frac{1}{\delta(y)^s}. \quad (2.19)$$

Proof of Theorem 1.4. We split the proof by building the solution associated to each datum f , g and h separately.

First case: $f, h \equiv 0$. We present here a readaptation of [60, Lemma 1.13].

The function u defined by equation (1.12) is continuous in Ω as an application of the Dominated Convergence Theorem and inequality (2.19). The continuity up to the boundary is postponed to Paragraph 2.4.2.

For the sake of clarity we divide the proof in four steps: for special forms of g , for g regular enough, for g bounded and finally for any other g .

Step 1. Suppose we have a measure ν , such that $\nu(\Omega) = 0$ and

$$g(x) = \int_{\mathcal{C}\Omega} \Gamma_s(x - y) d\nu(y) \quad x \in \mathcal{C}\Omega.$$

Then set

$$\tilde{u}(x) = \int_{\mathbb{R}^N} \Gamma_s(x - y) d\nu(y) \quad \text{for any } x \in \mathbb{R}^N :$$

a) $\tilde{u} = g$ in $\mathcal{C}\Omega$, since ν is supported in $\mathcal{C}\Omega$,

b) $\tilde{u} = u$ in Ω , where u is given by

$$u(x) = - \int_{\mathcal{C}\Omega} g(y) (-\Delta)^s G_\Omega(x, y) dy, \quad x \in \Omega,$$

indeed,

$$\begin{aligned} - \int_{\mathcal{C}\Omega} g(y) \cdot (-\Delta)^s G_\Omega(x, y) dy &= - \int_{\mathcal{C}\Omega} (-\Delta)^s G_\Omega(x, y) \left(\int_{\mathcal{C}\Omega} \Gamma_s(z - y) d\nu(z) \right) dy \\ &= - \int_{\mathcal{C}\Omega} \left(\int_{\mathcal{C}\Omega} \Gamma_s(z - y) \cdot (-\Delta)^s G_\Omega(x, y) dy \right) d\nu(z) \\ &\quad [\text{in view of equation (2.9)}] \\ &= \int_{\mathcal{C}\Omega} \Gamma_s(z - x) d\nu(z) = \tilde{u}(x), \end{aligned}$$

c) u is s -harmonic in Ω :

$$\begin{aligned} (\eta_r * u)(x) &= \int_{\mathcal{C}B_r(x)} \eta_r(y - x) \cdot u(y) dy = \int_{\mathcal{C}B_r(x)} \eta_r(y - x) \left(\int_{\mathbb{R}^N} \Gamma_s(z - y) d\nu(z) \right) dy \\ &= \int_{\mathbb{R}^N} \left(\int_{\mathcal{C}B_r(x)} \Gamma_s(z - y) \eta_r(y - x) dy \right) d\nu(z) = \int_{\mathbb{R}^N} \Gamma_s(x - z) d\nu(z) = u(x) \end{aligned}$$

by choosing $0 < r < \delta(x) \leq |z - x|$ and exploiting the s -harmonicity of Γ_s .

Step 2. If $g \in C^\infty(\overline{\mathcal{C}\Omega})$ and $\text{supp } g$ is bounded, then g admits an extension $\tilde{g} \in C_c^\infty(\mathbb{R}^N)$ and (see [60, Lemma 1.1]) there exists an absolutely continuous measure ν , with density Ψ , such that

$$\tilde{g}(x) = \int_{\mathcal{C}\Omega} \Gamma_s(x-y) d\nu(y) \quad \text{for any } x \in \mathbb{R}^N.$$

Denote by ν_Ω the measure obtained by restricting ν to Ω , i.e. $\nu_\Omega(A) = \nu(A \cap \Omega)$ for any measurable A and ν_Ω has density $\Psi_\Omega = \Psi \chi_\Omega$, and

$$\nu'_\Omega(y) = \begin{cases} \int_\Omega (-\Delta)^s G_\Omega(x, y) d\nu_\Omega(x) = - \int_\Omega (-\Delta)^s G_\Omega(x, y) \Psi_\Omega(x) dx & y \in \mathcal{C}\Omega \\ 0 & y \in \Omega \end{cases} :$$

the integral is well-defined because $\Psi_\Omega \in L^1(\Omega)$ while $(-\Delta)^s G_\Omega(\cdot, y) \in C(\overline{\Omega})$ for any fixed $y \in \mathcal{C}\Omega$ as a consequence of $G_\Omega(\cdot, y) \in C(\overline{\Omega})$ and equation (2.8). Define $\gamma = \nu - \nu_\Omega + \nu'_\Omega$ which is a measure supported in $\mathcal{C}\Omega$. Then, when $x \in \mathcal{C}\Omega$,

$$\int_{\mathbb{R}^N} \Gamma_s(x-y) d\nu'_\Omega(y) = - \int_\Omega \left(\int_{\mathcal{C}\Omega} \Gamma_s(x-y) (-\Delta)^s G_\Omega(z, y) dy \right) d\nu_\Omega(z) = \int_\Omega \Gamma_s(x-z) d\nu_\Omega(z)$$

where we have used (2.9). Therefore

$$\int_{\mathcal{C}\Omega} \Gamma_s(x-y) d\gamma(y) = \int_{\mathcal{C}\Omega} \Gamma_s(x-y) d\nu(y) = \tilde{g}(x)$$

so that we can apply the previous step of the proof.

Step 3. For $g \in L^\infty(\mathcal{C}\Omega)$, consider a sequence $\{g_n\}_{n \in \mathbb{N}} \subseteq C^\infty(\overline{\mathcal{C}\Omega})$ uniformly bounded and converging pointwisely to g . The corresponding sequence of s -harmonic functions u_n converges to u , since

$$u_n(x) = - \int_{\mathcal{C}\Omega} g_n(y) (-\Delta)^s G_\Omega(x, y) dy \xrightarrow{n \uparrow +\infty} u(x) = - \int_{\mathcal{C}\Omega} g(y) (-\Delta)^s G_\Omega(x, y) dy$$

by Dominated Convergence. Then, again by the Dominated Convergence theorem we have

$$u_n = \eta_\delta * u_n \longrightarrow \eta_\delta * u, \quad \text{in } \Omega$$

therefore

$$u = \lim u_n = \eta_\delta * u, \quad \text{in } \Omega,$$

i.e. u is s -harmonic in Ω .

Step 4. For a general measurable nonnegative g it suffices now to consider an increasing sequence g_n converging to g , e.g. $g_n = \min\{g, n\}$. Then the corresponding sequence of s -harmonic functions u_n converges to u . Moreover, the sequence $\{u_n\}_{n \in \mathbb{N}}$ is increasing:

$$u_{n+1}(x) = - \int_{\mathcal{C}\Omega} g_{n+1}(y) (-\Delta)^s G_\Omega(x, y) dy \geq - \int_{\mathcal{C}\Omega} g_n(y) (-\Delta)^s G_\Omega(x, y) dy = u_n(x).$$

Then, thanks to the Monotone Convergence theorem we have

$$u_n = \eta_\delta * u_n \longrightarrow \eta_\delta * u, \quad \text{in } \Omega$$

therefore

$$u = \lim u_n = \eta_\delta * u, \quad \text{in } \Omega.$$

Uniqueness. Finally, if $g \in C(\overline{\mathcal{C}\Omega})$, the solution we have built is the only solution in $C(\overline{\Omega})$ as an application of Lemma 2.8.

REMARK 2.9 | Suppose to have $g : \mathcal{C}\Omega \rightarrow [0, +\infty)$ for which (2.18) fails and there is a set $\mathcal{O} \subseteq \Omega$, $|\mathcal{O}| > 0$, in which

$$- \int_{\mathcal{C}\Omega} g(y) (-\Delta)^s G_\Omega(x, y) dy = +\infty, \quad x \in \mathcal{O}.$$

It is not possible in this case to have a pointwise solution of

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega, \\ u = g & \text{in } \mathcal{C}\Omega. \end{cases}$$

Indeed, if we set $g_n = \min\{g, n\}$, $n \in \mathbb{N}$, then g_n converges monotonically to g , and

$$u(x) \geq - \int_{\mathcal{C}\Omega} g_n(y) (-\Delta)^s G_\Omega(x, y) dy \xrightarrow{n \uparrow +\infty} +\infty$$

for all $x \in \mathcal{O}$.

Regularity. As a last step we notice that the C^∞ regularity is a consequence of the representation on balls and the regularity of the associated Poisson kernel therein. Properties listed in (1.8) are a consequence of Paragraph 5.3 below and the estimates on $(-\Delta)^s G_\Omega(x, y)$, $x \in \Omega$, $y \in \mathcal{C}\Omega$.

Second case: $g, h \equiv 0$. Use the construction of G_Ω to write for $x \in \Omega$

$$u(x) = \int_{\Omega} f(y) G_\Omega(y, x) dy = \int_{\Omega} f(y) \Gamma_s(y - x) dy - \int_{\Omega} f(y) H(y, x) dy :$$

the first addend is a function $u_1(x)$ which solves $(-\Delta)^s u_1 = f \chi_\Omega$ in \mathbb{R}^N , let us turn to the second one:

$$\begin{aligned} u_2(x) &:= \int_{\Omega} f(y) H(y, x) dy = - \int_{\Omega} f(y) \left[\int_{\mathcal{C}\Omega} \Gamma_s(y - z) (-\Delta)^s H(x, z) dz \right] dy \\ &= - \int_{\mathcal{C}\Omega} (-\Delta)^s H(x, z) \left[\int_{\Omega} f(y) \Gamma_s(y - z) dy \right] dz \\ &= - \int_{\mathcal{C}\Omega} (-\Delta)^s H(x, z) u_1(z) dz = - \int_{\mathcal{C}\Omega} u_1(z) (-\Delta)^s G_\Omega(x, z) dz. \end{aligned}$$

According to the *Step 4* above, u_2 solves

$$\begin{cases} (-\Delta)^s u_2 = 0 & \text{in } \Omega \\ u_2 = u_1 & \text{in } \mathcal{C}\Omega \end{cases}$$

therefore $u = u_1 - u_2$ and

$$\begin{cases} (-\Delta)^s u = (-\Delta)^s u_1 - (-\Delta)^s u_2 = f & \text{in } \Omega \\ u = u_1 - u_2 = 0 & \text{in } \mathcal{C}\Omega. \end{cases} \quad (2.20)$$

Finally, $u \in C^{2s+\alpha}(\Omega)$ thanks to [78, Proposition 2.8], while for inequality

$$|u(x)| \leq C \|f\|_\infty \delta(x)^s$$

we refer to [73, Proposition 1.1].

REMARK 2.10 | Note that these computations give an alternative integral representation to the one provided in equation (2.7) for u , meaning that we have both

$$\int_{\Omega} G_\Omega(x, y) (-\Delta)^s u(y) dy = u(x) = \int_{\Omega} G_\Omega(y, x) (-\Delta)^s u(y) dy,$$

so we must conclude that $G_\Omega(x, y) = G_\Omega(y, x)$ for $x, y \in \Omega$, $x \neq y$.

Third case: $f, g \equiv 0$. The function $u(x) = \int_{\partial\Omega} M_\Omega(x, \theta) h(\theta) d\mathcal{H}(\theta)$ is s -harmonic: to show this we use both the construction of $M_\Omega(x, \theta)$ and the mean value formula (2.1). Using the notations of (2.14), for any $0 < r < \delta(x)$ there exists $z \in \overline{B_r(x)}$

$$u_\varepsilon(x) = \int_{CB_r(x)} \eta_r(y) u_\varepsilon(x - y) dy + \gamma(N, s, r)(-\Delta)^s u_\varepsilon(z) = \int_{CB_r(x)} \eta_r(y) u_\varepsilon(x - y) dy + \gamma(N, s, r) f_\varepsilon(z)$$

where the equality $(-\Delta)^s u_\varepsilon = f_\varepsilon$ holds throughout Ω . Letting $\varepsilon \downarrow 0$ we have both

$$u_\varepsilon(x) \xrightarrow{\varepsilon \downarrow 0} u(x) = \int_{\partial\Omega} M_\Omega(x, \theta) h(\theta) d\mathcal{H}(\theta)$$

and

$$f_\varepsilon(z) \xrightarrow{\varepsilon \downarrow 0} 0, \quad \text{for any } z \in \Omega.$$

If now we note that, for any $p \in [1, N/(N - 2s))$ it holds by the Jensen's Inequality

$$\int_{\Omega} |u_\varepsilon|^p \leq \|f_\varepsilon\|_{L^1(\Omega)}^{p-1} \int_{\Omega} \int_{\Omega} G_\Omega(x, y)^p dx dy$$

which is a quantity bounded independently of ε (see also Theorem 2.19 below), we deduce that $\{u_\varepsilon\}_\varepsilon$ is an equibounded and equicontinuous family in $L^1(\Omega)$ and so the convergence $u_\varepsilon \rightarrow u$ holds in $L^1(\Omega)$. Moreover, when x is far from $\partial\Omega$ and $\varepsilon < \delta(x)$

$$u_\varepsilon(x) = \int_{\Omega} G_\Omega(x, y) f_\varepsilon(y) dy \leq C \int_{\{\delta(y) < \varepsilon\}} \delta(y) f_\varepsilon(y) dy$$

which again is bounded independently on ε . So we have boundedness in $L_{loc}^\infty(\Omega)$ and convergence in $L^1(\Omega)$. This implies that we have equality

$$u(x) = \int_{CB_r} \eta_r(y) u(x - y) dy,$$

i.e. u is s -harmonic.

The fact that $\delta^{1-s}u \in C(\overline{\Omega})$ is done in Lemma 2.7, while the C^∞ regularity can be done once again with the representation on balls. \square

LEMMA 2.11: AN EXPLICIT EXAMPLE ON THE BALL | *If $0 < \sigma < 1 - s$, the functions*

$$u_\sigma(x) = \begin{cases} \frac{c(N, s)}{(1 - |x|^2)^\sigma} & |x| < 1 \\ \frac{c(N, s + \sigma)}{(|x|^2 - 1)^\sigma} & |x| > 1 \end{cases} \quad u_{1-s}(x) = \begin{cases} \frac{c(N, s)}{(1 - |x|^2)^{1-s}} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

are s -harmonic in the ball $B = B_1(0)$, where $c(N, s)$ is given by (1.4). Moreover in this case

$$M_B(x, \theta) = \lim_{\substack{y \in \Omega \\ y \rightarrow \theta \in \partial\Omega}} \frac{G_B^s(x, y)}{\delta(y)^s} = \frac{C(1 - |x|^2)^s}{|x - \theta|^N} \quad \text{for some } C = C(N, s).$$

Proof. According to [60, equation (1.6.11')] and in view to the computations due to Riesz [72], the Poisson kernel for the ball B of radius 1 and centered at 0 has the explicit expression

$$-(-\Delta)^s G_B^s(x, y) = \frac{c(N, s)}{|x - y|^N} \left(\frac{1 - |x|^2}{|y|^2 - 1} \right)^s, \quad c(N, s) = \frac{\Gamma(N/2) \sin(\pi s)}{\pi^{1+N/2}}. \quad (2.21)$$

We construct here the s -harmonic function induced by data

$$h(\theta) = 0, \quad g_\sigma(y) = \frac{c(N, s + \sigma)}{(|y|^2 - 1)^\sigma}, \quad 0 < \sigma < 1 - s.$$

Indeed, it can be explicitly computed

$$\begin{aligned}
(1 - |x|^2)^\sigma u_\sigma(x) &= \\
&= - (1 - |x|^2)^\sigma \int_{CB} g_\sigma(y) \cdot (-\Delta)^s G_B^s(x, y) dy = \int_{CB} \frac{c(N, s)}{|x - y|^N} \cdot \frac{(1 - |x|^2)^{s+\sigma}}{(|y|^2 - 1)^s} g_\sigma(y) dy = \\
&= \int_{CB} \frac{c(N, s) c(N, s + \sigma)}{|x - y|^N} \cdot \frac{(1 - |x|^2)^{s+\sigma}}{(|y|^2 - 1)^{s+\sigma}} dy = -c(N, s) \int_{CB} (-\Delta)^{s+\sigma} G_B^{s+\sigma}(x, y) dy = c(N, s)
\end{aligned}$$

therefore the function

$$u_\sigma(x) = \begin{cases} \frac{c(N, s)}{(1 - |x|^2)^\sigma} & x \in B \\ \frac{c(N, s + \sigma)}{(|x|^2 - 1)^\sigma} & x \in CB \end{cases}$$

solves the problem

$$\begin{cases} (-\Delta)^s u_\sigma = 0 & \text{in } B \\ u_\sigma(x) = g_\sigma(x) = \frac{c(N, s + \sigma)}{(|x|^2 - 1)^\sigma} & \text{in } CB. \end{cases}$$

We are interested in letting $\sigma \rightarrow 1 - s$. Obviously,

$$u_\sigma(x) \xrightarrow{\sigma \rightarrow 1-s} u(x) = \begin{cases} \frac{c(N, s)}{(1 - |x|^2)^{1-s}} & \text{in } B \\ 0 & \text{in } CB \end{cases}$$

everywhere in $\mathbb{R}^N \setminus \partial B$. The s -harmonicity is preserved, since for $x \in B$ and any $r \in (0, 1 - |x|)$,

$$\begin{aligned}
u(x) &= \lim_{\sigma \rightarrow 1-s} u_\sigma(x) = \lim_{\sigma \rightarrow 1-s} \int_{CB_r} u_\sigma(y) \eta_r(x - y) dy = \\
&= \lim_{\sigma \rightarrow 1-s} \int_{CB_r \cap B} u_\sigma(y) \eta_r(x - y) dy + \lim_{\sigma \rightarrow 1-s} \int_{CB} g_\sigma(y) \eta_r(x - y) dy = \\
&= \int_{CB_r \cap B} u(y) \eta_r(x - y) dy = \int_{CB_r} u(y) \eta_r(x - y) dy
\end{aligned}$$

since $g_\sigma \xrightarrow{\sigma \rightarrow 1-s} 0$ in $L^1(CB)$, while $\eta_r(\cdot - x)$ is bounded in CB for $0 < r < \delta(x)$. For any $\psi \in C_c^\infty(B)$ we have

$$\begin{aligned}
\int_B u_\sigma \psi &= - \int_{CB} g_\sigma(x) \int_B \psi(z) (-\Delta)^s G_B^s(z, x) dz dx = \\
&= - \int_{CB} \frac{c(N, s + \sigma)}{(|x|^2 - 1)^\sigma} \int_B \psi(z) (-\Delta)^s G_B^s(z, x) dz dx.
\end{aligned}$$

Then on the one hand

$$\int_B u_\sigma \psi \xrightarrow{\sigma \rightarrow 1-s} \int_B u \psi = c(N, s) \int_B \frac{\psi(x)}{(1 - |x|^2)^{1-s}} dx.$$

On the other hand

$$\begin{aligned}
&- \int_{CB} \frac{c(N, s + \sigma)}{(|x|^2 - 1)^\sigma} \int_B \psi(z) (-\Delta)^s G_B^s(z, x) dz dx = \\
&= \int_{CB} \frac{c(N, s + \sigma)}{(|x|^2 - 1)^\sigma} \left[\int_B \psi(z) \frac{c(N, s)}{|z - x|^N} \left(\frac{1 - |z|^2}{|x|^2 - 1} \right)^s dz \right] dx = \\
&= \int_{CB} \frac{c(N, s + \sigma) c(N, s)}{(|x|^2 - 1)^{\sigma+s}} \int_B \left[\psi(z) \frac{(1 - |z|^2)^s}{|z - x|^N} dz \right] dx.
\end{aligned}$$

Note that the function $\int_B \psi(z) \cdot \frac{(1-|z|^2)^s}{|z-x|^N} dz$ is $C(\overline{CB})$ in the x variable, since $\psi \in C_c^\infty(B)$. Splitting $x \in CB$ in spherical coordinates, i.e. $x = \rho\theta$, $\rho = |x| \in (1, +\infty)$ and $|\theta| = 1$, and denoting by $\phi \in C(\overline{\Omega})$ the function satisfying $(-\Delta)^s \phi|_\Omega = \psi$, $\phi = 0$ in $C\Omega$,

$$-c(N, s + \sigma) \int_{CB} \frac{(-\Delta)^s \phi(x)}{(|x|^2 - 1)^\sigma} dx = \int_1^{+\infty} \frac{c(N, s + \sigma) c(N, s)}{(\rho^2 - 1)^{\sigma+s}} \cdot \psi_1(\rho) \rho^{N-1} d\rho \quad (2.22)$$

where $\psi_1(\rho) = \int_{\partial B} \int_B \psi(z) \frac{(1-|z|^2)^s}{|z-\rho\theta|^N} dz d\mathcal{H}(\theta)$ is continuous on $[1, +\infty)$ and has a decay at infinity which is comparable to that of ρ^{-N} . Therefore, as $\sigma \rightarrow 1 - s$,

$$\begin{aligned} -c(N, s + \sigma) \int_{CB} \frac{(-\Delta)^s \phi(x)}{(|x|^2 - 1)^\sigma} dx &\longrightarrow c(N, s) c(N, 1/2) \frac{\psi_1(1)}{2} = \\ &= \frac{c(N, s) c(N, 1/2)}{2} \int_{\partial B} \int_B \psi(z) \cdot \frac{(1-|z|^2)^s}{|z-\theta|^N} dz d\mathcal{H}(\theta). \end{aligned}$$

Indeed, by the definition of $c(N, s)$ in (1.4), it holds

$$-c(N, s + \sigma) \int_1^{+\infty} \frac{\rho d\rho}{(\rho^2 - 1)^{\sigma+s}} = -\frac{c(N, s + \sigma)}{2(1 - s - \sigma)} \xrightarrow{\sigma \uparrow 1-s} \frac{c(N, 1/2)}{2\pi}$$

and, since in (2.22) the product $\psi_1(\rho) \rho^{N-2} \in C([1, +\infty))$,

$$-c(N, s + \sigma) \int_{CB} \frac{(-\Delta)^s \phi(x)}{(|x|^2 - 1)^\sigma} dx \longrightarrow c(N, s) c(N, 1/2) \frac{\psi_1(1)}{2}.$$

So

$$\int_B u\psi = \frac{c(N, s) c(N, 1/2)}{2} \int_{\partial B} \int_B \psi(z) \cdot \frac{(1-|z|^2)^s}{|z-\theta|^N} dz d\mathcal{H}(\theta),$$

i.e.

$$u(x) = \frac{c(N, s) c(N, 1/2)}{2} \int_{\partial B} \frac{(1-|x|^2)^s}{|x-\theta|^N} d\mathcal{H}(\theta)$$

and then the kernel $M_B(x, \theta)$ for the ball is

$$M_B(x, \theta) = \lim_{\substack{y \in \Omega \\ y \rightarrow \theta \in \partial\Omega}} \frac{G_B^s(x, y)}{\delta(y)^s} = C \cdot \frac{(1-|x|^2)^s}{|x-\theta|^N}$$

where $C = C(N, s)$. □

2.3 THE LINEAR DIRICHLET PROBLEM: AN L^1 THEORY

We define L^1 solutions for the Dirichlet problem, in the spirit of Stampacchia [83]. A proper functional space in which to consider test functions is the following.

LEMMA 2.12: TEST FUNCTION SPACE | For any $\psi \in C_c^\infty(\Omega)$ the solution ϕ of

$$\begin{cases} (-\Delta)^s \phi = \psi & \text{in } \Omega \\ \phi = 0 & \text{in } C\Omega \\ E\phi = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies the following

1. $(-\Delta)^s \phi \in L^1(\mathcal{C}\Omega)$ and for any $x \in \mathcal{C}\Omega$

$$(-\Delta)^s \phi(x) = \int_{\Omega} \psi(z) (-\Delta)^s G_{\Omega}(z, x) dz, \quad (2.23)$$

2. for any $\theta \in \partial\Omega$

$$\int_{\Omega} M_{\Omega}(x, \theta) \psi(x) dx = D_s \phi(\theta). \quad (2.24)$$

Proof. The solution ϕ is given by $\phi(x) = \int_{\mathbb{R}^N} \psi(y) G_{\Omega}(y, x) dy \in C^{2s+\varepsilon}(\Omega) \cap C(\overline{\Omega})$. Also, when $x \in \mathcal{C}\Omega$,

$$\begin{aligned} (-\Delta)^s \phi(x) &= -C_{N,s} \int_{\Omega} \frac{\phi(y)}{|y-x|^{N+2s}} dy = -\int_{\Omega} \frac{C_{N,s}}{|y-x|^{N+2s}} \left[\int_{\Omega} \psi(z) G_{\Omega}(z, y) dz \right] dy = \\ &= -\int_{\Omega} \psi(z) \left[C_{N,s} \int_{\Omega} \frac{G_{\Omega}(z, y)}{|y-x|^{N+2s}} dy \right] dz = \int_{\Omega} \psi(z) (-\Delta)^s G_{\Omega}(z, x) dz. \end{aligned}$$

Thanks to the integrability of $(-\Delta)^s G_{\Omega}(z, x)$ in $\mathcal{C}\Omega$, also $(-\Delta)^s \phi \in L^1(\mathcal{C}\Omega)$, so that ϕ is an admissible function for the integration by parts formula (1.7). Moreover,

$$\int_{\Omega} M_{\Omega}(x, \theta) \psi(x) dx = D_s \phi(\theta).$$

Indeed,

$$\int_{\Omega} M_{\Omega}(x, \theta) \psi(x) dx = \int_{\Omega} \psi(x) \lim_{\substack{z \in \Omega \\ z \rightarrow \theta}} \frac{G_{\Omega}(x, z)}{\delta(z)^s} dx = \lim_{\substack{z \in \Omega \\ z \rightarrow \theta}} \int_{\Omega} \frac{\psi(x) G_{\Omega}(x, z)}{\delta(z)^s} dx = \lim_{\substack{z \in \Omega \\ z \rightarrow \theta}} \frac{\phi(z)}{\delta(z)^s} = D_s \phi(\theta).$$

□

Then, our space of test functions will be

$$\mathcal{T}(\Omega) = \left\{ \phi \in C(\mathbb{R}^N) : \begin{cases} (-\Delta)^s \phi = \psi & \text{in } \Omega \\ \phi = 0 & \text{in } \mathcal{C}\Omega \\ E\phi = 0 & \text{in } \partial\Omega \end{cases}, \text{ when } \psi \in C_c^\infty(\Omega) \right\}.$$

Note that the map D_s is well-defined from $\mathcal{T}(\Omega)$ to $C(\partial\Omega)$ as a consequence of the results in [73, Theorem 1.2]. We give the following

DEFINITION 2.13: WEAK L^1 SOLUTION | Given three Radon measures $\lambda \in \mathcal{M}(\Omega)$, $\mu \in \mathcal{M}(\mathcal{C}\Omega)$ and $\nu \in \mathcal{M}(\partial\Omega)$, we say that a function $u \in L^1(\Omega)$ is a solution of

$$\begin{cases} (-\Delta)^s u = \lambda & \text{in } \Omega \\ u = \mu & \text{in } \mathcal{C}\Omega \\ Eu = \nu & \text{on } \partial\Omega \end{cases}$$

if for every $\phi \in \mathcal{T}(\Omega)$ it holds

$$\int_{\Omega} u(x) (-\Delta)^s \phi(x) dx = \int_{\Omega} \phi(x) d\lambda(x) - \int_{\mathcal{C}\Omega} (-\Delta)^s \phi(x) d\mu(x) + \int_{\partial\Omega} D_s \phi(\theta) d\nu(\theta) \quad (2.25)$$

where $D_s \phi(\theta) = \lim_{\substack{x \in \Omega \\ x \rightarrow \theta}} \frac{\phi(x)}{\delta(x)^s}$.

PROPOSITION 2.14 | Solutions provided by Theorem 1.4 are solutions in the above L^1 sense.

Proof. Indeed, if

$$u(x) = \int_{\Omega} f(y) G_{\Omega}(y, x) dy - \int_{C\Omega} g(y) (-\Delta)^s G_{\Omega}(x, y) dy + \int_{\partial\Omega} M_{\Omega}(x, \theta) h(\theta) d\mathcal{H}(\theta),$$

then for any $\phi \in \mathcal{T}(\Omega)$,

$$\begin{aligned} \int_{\Omega} u(x) (-\Delta)^s \phi(x) dx &= \\ &= \int_{\Omega} \left[\int_{\Omega} f(y) G_{\Omega}(y, x) dy - \int_{C\Omega} g(y) (-\Delta)^s G_{\Omega}(x, y) dy + \int_{\partial\Omega} M_{\Omega}(x, \theta) h(\theta) d\mathcal{H}(\theta) \right] (-\Delta)^s \phi(x) dx \\ &= \int_{\Omega} f(y) \int_{\Omega} (-\Delta)^s \phi(x) G_{\Omega}(y, x) dx dy - \int_{C\Omega} g(y) \int_{\Omega} (-\Delta)^s \phi(x) (-\Delta)^s G_{\Omega}(x, y) dx dy + \\ &\quad + \int_{\partial\Omega} h(\theta) \int_{\Omega} M_{\Omega}(x, \theta) (-\Delta)^s \phi(x) d\mathcal{H}(\theta). \end{aligned}$$

Apply now (2.23) and (2.24) to deduce the thesis. \square

Proof of Theorem 1.9. In case $\nu = 0$, we claim that the solution is given by formula

$$u(x) = \int_{\Omega} G_{\Omega}(y, x) d\lambda(y) - \int_{C\Omega} (-\Delta)^s G_{\Omega}(x, y) d\mu(y).$$

Take $\phi \in \mathcal{T}(\Omega)$:

$$\begin{aligned} \int_{\Omega} u(x) (-\Delta)^s \phi(x) dx &= \int_{\Omega} \left[\int_{\Omega} G_{\Omega}(y, x) d\lambda(y) - \int_{C\Omega} (-\Delta)^s G_{\Omega}(x, y) d\mu(y) \right] (-\Delta)^s \phi(x) dx = \\ &= \int_{\Omega} \left[\int_{\Omega} (-\Delta)^s \phi(x) G_{\Omega}(y, x) dx \right] d\lambda(y) - \int_{C\Omega} \left[\int_{\Omega} (-\Delta)^s \phi(x) (-\Delta)^s G_{\Omega}(x, y) dx \right] d\mu(y) = \\ &= \int_{\Omega} \phi(y) d\lambda(y) - \int_{C\Omega} (-\Delta)^s \phi(y) d\mu(y) \end{aligned}$$

again using (2.23). Then, we claim that the function

$$u(x) = \int_{\partial\Omega} M_{\Omega}(x, \theta) d\nu(\theta)$$

solves problem

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega, \\ u = 0 & \text{in } C\Omega, \\ Eu = \nu & \text{on } \partial\Omega. \end{cases}$$

This, along with the first part of the proof, proves our thesis. Take $\phi \in \mathcal{T}(\Omega)$ and call $\psi = (-\Delta)^s \phi|_{\Omega} \in C_c^{\infty}(\Omega)$. Then

$$\begin{aligned} \int_{\Omega} \left(\int_{\partial\Omega} M_{\Omega}(x, \theta) d\nu(\theta) \right) \psi(x) dx &= \int_{\partial\Omega} \left(\int_{\Omega} M_{\Omega}(x, \theta) \psi(x) dx \right) d\nu(\theta) \\ &= \int_{\partial\Omega} \left(\int_{\Omega} \psi(x) \lim_{\substack{z \in \Omega \\ z \rightarrow \theta}} \frac{G_{\Omega}(x, z)}{\delta(z)^s} dx \right) d\nu(\theta) \\ &= \int_{\partial\Omega} \lim_{\substack{z \in \Omega \\ z \rightarrow \theta}} \left(\int_{\Omega} \frac{\psi(x) G_{\Omega}(x, z)}{\delta(z)^s} dx \right) d\nu(\theta) \\ &= \int_{\partial\Omega} \lim_{\substack{z \in \Omega \\ z \rightarrow \theta}} \frac{\phi(z)}{\delta(z)^s} d\nu(\theta) = \int_{\partial\Omega} D_s \phi(\theta) d\nu(\theta). \end{aligned}$$

The uniqueness is due to Lemma 2.16 below. Theorem 2.17 below proves the estimate on the L^1 norm of the solution. \square

COROLLARY 2.15 | *The solution provided by Theorem 1.9 satisfies, for any open $A \subseteq \Omega$ such that $\overline{A} \subseteq \Omega$ and any $\phi_A \in \mathcal{T}(A)$,*

$$\int_A u(x)(-\Delta)^s \phi_A(x) dx = \int_A \phi_A(x) d\lambda(x) - \int_{\Omega \setminus A} u(x)(-\Delta)^s \phi_A(x) dx - \int_{\mathcal{C}\Omega} (-\Delta)^s \phi(x) d\mu(x).$$

Proof. For the sake of clarity we give the proof in the particular case $\mu = \nu = 0$; the other cases follow with the same computations with the necessary modifications. Establish problem

$$\begin{cases} (-\Delta)^s u_A = \lambda & \text{in } A \\ u_A = u & \text{in } \mathcal{C}A \\ Eu_A = 0 & \text{on } \partial A \end{cases} :$$

we want to prove that $u_A = u$ in A . The function u_A is given by

$$\begin{aligned} u_A(x) &= \int_A G_A(x, z) d\lambda(z) - \int_{\Omega \setminus A} (-\Delta)^s G_A(x, z) u(z) dz \\ &= \int_A G_A(x, z) d\lambda(z) - \int_{\Omega \setminus A} (-\Delta)^s G_A(x, z) \left[\int_{\Omega} G_{\Omega}(z, y) d\lambda(y) \right] dz \\ &= \int_A G_A(x, z) d\lambda(z) - \int_A \left[\int_{\Omega \setminus A} (-\Delta)^s G_A(x, y) G_{\Omega}(y, z) dy \right] d\lambda(z) \\ &\quad - \int_{\Omega \setminus A} \left[\int_{\Omega \setminus A} (-\Delta)^s G_A(x, y) G_{\Omega}(y, z) dy \right] d\lambda(z). \end{aligned}$$

Since now, by construction of Green functions,

$$\begin{aligned} G_A(x, z) - \int_{\Omega \setminus A} (-\Delta)^s G_A(x, y) G_{\Omega}(y, z) dy &= G_{\Omega}(x, z) \quad x, z \in A, \\ \int_{\Omega \setminus A} (-\Delta)^s G_A(x, y) G_{\Omega}(y, z) dy &= (-\Delta)^s G_{\Omega}(x, z) \quad x \in A, z \in \mathcal{C}A, \end{aligned}$$

we deduce $u_A(x) = u(x)$, $x \in A$. □

LEMMA 2.16: MAXIMUM PRINCIPLE | *Let $u \in L^1(\Omega)$ be a solution of*

$$\begin{cases} (-\Delta)^s u \leq 0 & \text{in } \Omega, \\ u \leq 0 & \text{in } \mathcal{C}\Omega, \\ Eu \leq 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \leq 0$ a.e. in \mathbb{R}^N .

Proof. Take $\psi \in C_c^\infty(\Omega)$, $\psi \geq 0$ and the associated $\phi \in \mathcal{T}(\Omega)$ for which $(-\Delta)^s \phi|_{\Omega} = \psi$: it is $\phi \geq 0$ in Ω , in view of Lemma 2.8. This implies that for $y \in \mathcal{C}\Omega$ it holds

$$(-\Delta)^s \phi(y) = -C_{N,s} \int_{\Omega} \frac{\phi(z)}{|y-z|^N} dz \leq 0,$$

and also $D_s \phi \geq 0$ throughout $\partial\Omega$. In particular

$$\int_{\Omega} u \psi \leq 0$$

as a consequence of (2.25). □

2.3.1 Regularity theory

THEOREM 2.17 | Given three Radon measures $\lambda \in \mathcal{M}(\Omega)$, $\mu \in \mathcal{M}(\mathcal{C}\Omega)$ and $\nu \in \mathcal{M}(\partial\Omega)$, consider the solution $u \in L^1(\Omega)$ of problem

$$\begin{cases} (-\Delta)^s u = \lambda & \text{in } \Omega, \\ u = \mu & \text{in } \mathcal{C}\Omega, \\ Eu = \nu & \text{on } \partial\Omega. \end{cases}$$

Then

$$\|u\|_{L^1(\Omega)} \leq C \left(\|\lambda\|_{\mathcal{M}(\Omega, \delta(x)^s dx)} + \|\mu\|_{\mathcal{M}(\mathcal{C}\Omega, \delta(x)^{-s} \wedge \delta(x)^{-n-2s} dx)} + \|\nu\|_{\mathcal{M}(\partial\Omega)} \right).$$

Proof. Consider ζ to be the solution of

$$\begin{cases} (-\Delta)^s \zeta = 1 & \text{in } \Omega \\ \zeta = 0 & \text{in } \mathcal{C}\Omega \\ E\zeta = 0 & \text{on } \partial\Omega \end{cases}$$

which we know to satisfy $0 \leq \zeta(x) \leq C \delta(x)^s$ in Ω , see [73]. Note also that, by approximating ζ with functions in $\mathcal{T}(\Omega)$ and by (2.23), for $x \in \mathcal{C}\Omega$

$$(-\Delta)^s \zeta(x) = \int_{\Omega} (-\Delta)^s G_{\Omega}(z, x) dz, \quad x \in \mathcal{C}\Omega,$$

and therefore, when $x \in \mathcal{C}\Omega$ and $\delta(x) < 1$,

$$0 \leq -(-\Delta)^s \zeta(x) \leq \int_{\Omega} \frac{C \delta(y)^s}{|y-x|^{N+2s}} dy \leq C \int_{\delta(x)}^{+\infty} \frac{dt}{t^{1+s}} = \frac{C}{\delta(x)^s},$$

while for $x \in \mathcal{C}\Omega$ and $\delta(x) \geq 1$

$$0 \leq -(-\Delta)^s \zeta(x) \leq \int_{\Omega} \frac{C \delta(y)^s}{|y-x|^{N+2s}} dy \leq \frac{C}{\delta(x)^{N+2s}} \int_{\Omega} \delta(y)^s dy.$$

Furthermore, $D_s \zeta$ is a well-defined function on $\partial\Omega$, again thanks to the results in [73, Proposition 1.1]. Indeed, consider an increasing sequence $\{\psi_k\}_{k \in \mathbb{N}} \subseteq C_c^\infty(\Omega)$, such that $0 \leq \psi_k \leq 1$, $\psi_k \uparrow 1$ in Ω . Call ϕ_k the function in $\mathcal{T}(\Omega)$ associated with ψ_k , i.e. $(-\Delta)^s \phi_k|_{\Omega} = \psi_k$. In this setting

$$0 \leq \lim_{\substack{z \in \Omega \\ z \rightarrow \theta}} \frac{\zeta(z) - \phi_k(z)}{\delta(z)^s} = \lim_{\substack{z \in \Omega \\ y \rightarrow \theta}} \int_{\Omega} \frac{G_{\Omega}(y, z)}{\delta(z)^s} (1 - \psi_k(y)) dy \leq \int_{\Omega} M_{\Omega}(y, \theta) (1 - \psi_k(y)) dy \xrightarrow{k \rightarrow +\infty} 0,$$

hence $D_s \zeta(\theta)$ is well-defined and it holds

$$D_s \zeta(\theta) = \int_{\Omega} M_{\Omega}(x, \theta) dx.$$

Finally, we underline how $D_s \zeta \in L^\infty(\Omega)$, thanks to (2.17).

We split the rest of the proof by using the integral representation of u :

i the mass induced by the right-hand side is

$$\int_{\Omega} \left| \int_{\Omega} G_{\Omega}(y, x) d\lambda(y) \right| dx \leq \int_{\Omega} \int_{\Omega} G_{\Omega}(y, x) dx d|\lambda|(y) \leq \int_{\Omega} \zeta(y) d|\lambda|(y) \leq C \int_{\Omega} \delta(y)^s d|\lambda|(y),$$

ii the one induced by the external datum

$$\begin{aligned} \int_{\Omega} \left| \int_{\mathcal{C}\Omega} (-\Delta)^s G_{\Omega}(x, y) d\mu(y) \right| dx &\leq \int_{\mathcal{C}\Omega} - \int_{\Omega} (-\Delta)^s G_{\Omega}(x, y) dx d|\mu|(y) = \\ &= \int_{\mathcal{C}\Omega} -(-\Delta)^s \zeta(y) d|\mu|(y) \leq \int_{\mathcal{C}\Omega} \left(\frac{C}{\delta(y)^s} \wedge \frac{C}{\delta(y)^{N+2s}} \right) d|\mu|(y), \end{aligned}$$

iii finally the mass due to the boundary behavior

$$\begin{aligned} \int_{\Omega} \left| \int_{\partial\Omega} M_{\Omega}(x, \theta) d\nu(\theta) \right| dx &\leq \int_{\partial\Omega} \int_{\Omega} M_{\Omega}(x, \theta) dx d|\nu|(\theta) = \\ &= \int_{\partial\Omega} D_s \zeta(\theta) d|\nu|(\theta) \leq \|D_s \zeta\|_{\infty} |\nu|(\partial\Omega). \end{aligned}$$

Note that the smoothness of the domain is needed only to make the last point go through, and we can repeat the proof in case $\nu = 0$ without requiring $\partial\Omega \in C^{1,1}$. \square

To gain higher integrability on a solution, the first step we take is the following

LEMMA 2.18 | *For any $q > N/s$ there exists $C = C(N, s, \Omega, q)$ such that for all $\phi \in \mathcal{T}(\Omega)$*

$$\left\| \frac{\phi}{\delta^s} \right\|_{L^{\infty}(\Omega)} \leq C \|(-\Delta)^s \phi\|_{L^q(\Omega)}.$$

Proof. Call as usual $\psi = (-\Delta)^s \phi|_{\Omega} \in C_c^{\infty}(\Omega)$. Let us work formally

$$\left\| \frac{\phi}{\delta^s} \right\|_{L^{\infty}(\Omega)} \leq \sup_{x \in \Omega} \frac{1}{\delta(x)^s} \int_{\Omega} G_{\Omega}(x, y) |\psi(y)| dy \leq \sup_{x \in \Omega} \frac{\|G_{\Omega}(x, \cdot)\|_{L^p(\Omega)}}{\delta(x)^s} \cdot \|(-\Delta)^s \phi\|_{L^q(\Omega)},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Thus we need only to understand for what values of p we are not writing a trivial inequality. The main tool here is inequality

$$G_{\Omega}(x, y) \leq \frac{c \delta(x)^s}{|x - y|^N} (|x - y|^s \wedge \delta(y)^s), \quad (2.26)$$

which holds in $C^{1,1}$ domains, see [27, equation 2.14]. We have then

$$\int_{\Omega} \left| \frac{G_{\Omega}(x, y)}{\delta(x)^s} \right|^p dy \leq c \int_{\Omega} \frac{1}{|x - y|^{Np}} (|x - y|^{sp} \wedge \delta(y)^{sp}) dy \leq c \int_{\Omega} \frac{1}{|x - y|^{(N-s)p}}$$

which is uniformly bounded in x for $p < N/(N - s)$. This condition on p becomes $q > N/s$. \square

In view of this last lemma, we are able to provide the following theorem, which is the fractional counterpart of a classical result (see e.g. [41, Proposition A.9.1]).

THEOREM 2.19 | *For any $p < N/(N - s)$, there exists a constant C such that any solution $u \in L^1(\Omega)$ of problem*

$$\begin{cases} (-\Delta)^s u = \lambda & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

has a finite L^p -norm controlled by

$$\|u\|_{L^p(\Omega)} \leq C \|\lambda\|_{L^1(\Omega, \delta(x)^s dx)}$$

Proof. For any $\psi \in C_c^{\infty}(\Omega)$, let $\phi \in \mathcal{T}(\Omega)$ be chosen in such a way that $(-\Delta)^s \phi|_{\Omega} = \psi$. We have

$$\left| \int_{\Omega} u(x) \psi(x) dx \right| = \left| \int_{\Omega} \phi(x) d\lambda(x) \right| \leq \int_{\Omega} \frac{|\phi(x)|}{\delta(x)^s} \cdot \delta(x)^s d|\lambda|(x) \leq C \|\psi\|_{L^q(\Omega)} \int_{\Omega} \delta(x)^s d|\lambda|(x).$$

where q is the conjugate exponent of p , according to Lemma 2.18. By density of $C_c^{\infty}(\Omega)$ in $L^q(\Omega)$ and the isometry between $L^p(\Omega)$ and the dual space of $L^q(\Omega)$, we obtain our thesis. \square

2.4 ASYMPTOTIC BEHAVIOUR AT THE BOUNDARY

2.4.1 Right-hand side blowing up at the boundary: proof of (1.16)

In this paragraph we study the boundary behaviour of the solution u to the problem

$$\begin{cases} (-\Delta)^s u(x) = \delta(x)^{-\beta} & \text{in } \Omega, \ 0 < \beta < 1 + s \\ u = 0 & \text{in } \mathcal{C}\Omega \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

starting from the radial case $\Omega = B$. Then

$$u(x) = \int_B \frac{G_B(x, y)}{\delta(y)^\beta} dy$$

and by using (1.14), up to multiplicative constants,

$$u(x) \leq \int_B \frac{[|x - y|^2 \wedge \delta(x)\delta(y)]^s}{|x - y|^N} \cdot \frac{dy}{\delta(y)^\beta}.$$

Set

$$\varepsilon := \delta(x) \text{ and } x = (1 - \varepsilon)e_1, \quad \theta := \frac{y}{|y|}, \ r := \delta(y) \text{ and } y = (1 - r)\theta,$$

and rewrite

$$u(x) \leq \int_{S^{N-1}} \int_0^1 \frac{[(1 - \varepsilon)e_1 - (1 - r)\theta]^2 \wedge \varepsilon r]^s}{|(1 - \varepsilon)e_1 - (1 - r)\theta|^N} \cdot \frac{(1 - r)^{N-1}}{r^\beta} dr d\mathcal{H}^{N-1}(\theta).$$

Split the angular variable in $\theta = (\theta_1, \theta')$ where $\theta_1 = \langle \theta, e_1 \rangle$ and $\theta_1^2 + |\theta'|^2 = 1$: then, for a general F ,

$$\int_{S^{N-1}} F(\theta) d\mathcal{H}^{N-1}(\theta) = \int_{S^{N-2}} \left(\int_{-1}^1 (1 - \theta_1^2)^{(N-3)/2} F(\theta_1, \theta') d\theta_1 \right) d\mathcal{H}^{N-2}(\theta'),$$

and we write

$$\begin{aligned} u(x) &\leq \int_{S^{N-2}} \int_0^1 (1 - \theta_1^2)^{(N-3)/2} \int_0^1 \frac{\{[(1 - \varepsilon)^2 + (1 - r)^2 - 2(1 - \varepsilon)(1 - r)\theta_1] \wedge \varepsilon r\}^s (1 - r)^{N-1}}{[(1 - \varepsilon)^2 + (1 - r)^2 - 2(1 - \varepsilon)(1 - r)\theta_1]^{N/2} r^\beta} dr d\theta_1 d\mathcal{H}^{N-2}(\theta') \\ &= \mathcal{H}^{N-2}(S^{N-2}) \int_0^1 (1 - \theta_1^2)^{(N-3)/2} \int_0^1 \frac{\{[(1 - \varepsilon)^2 + (1 - r)^2 - 2(1 - \varepsilon)(1 - r)\theta_1] \wedge \varepsilon r\}^s (1 - r)^{N-1}}{[(1 - \varepsilon)^2 + (1 - r)^2 - 2(1 - \varepsilon)(1 - r)\theta_1]^{N/2} r^\beta} dr d\theta_1. \end{aligned}$$

From now on we will drop all multiplicative constants and all inequalities will have to be interpreted to hold up to constants. Let us apply a first change of variables

$$t = \frac{1 - r}{1 - \varepsilon} \quad \longleftrightarrow \quad r = 1 - (1 - \varepsilon)t$$

to obtain

$$u(x) \leq \int_0^1 (1 - \theta_1^2)^{(N-3)/2} \int_0^{1/(1-\varepsilon)} \quad (2.27)$$

$$\begin{aligned} & \frac{\{[(1-\varepsilon)^2 + (1-\varepsilon)^2 t^2 - 2(1-\varepsilon)^2 t \theta_1] \wedge \varepsilon[1 - (1-\varepsilon)t]\}^s}{[(1-\varepsilon)^2 + (1-\varepsilon)^2 t^2 - 2(1-\varepsilon)^2 t \theta_1]^{N/2}} \frac{(1-\varepsilon)^N t^{N-1}}{(1 - (1-\varepsilon)t)^\beta} dt d\theta_1 \\ &= \int_0^1 (1 - \theta_1^2)^{(N-3)/2} \int_0^{1/(1-\varepsilon)} \frac{[(1-\varepsilon)^2(1+t^2 - 2t\theta_1) \wedge \varepsilon[1 - (1-\varepsilon)t]]^s}{[1+t^2 - 2t\theta_1]^{N/2}} \cdot \frac{t^{N-1}}{(1 - (1-\varepsilon)t)^\beta} dt d\theta_1 \\ &\leq \int_0^{1/(1-\varepsilon)} \int_0^1 \frac{\{[(1-t)^2 + 2t\sigma] \wedge \varepsilon[1 - (1-\varepsilon)t]\}^s}{[(1-t)^2 + 2t\sigma]^{N/2}} \sigma^{(N-3)/2} d\sigma \frac{t^{N-1}}{(1 - (1-\varepsilon)t)^\beta} dt, \end{aligned} \quad (2.28)$$

where $\sigma = 1 - \theta_1$. We compute now the integral in the variable σ . Let σ' be defined by equality

$$(1-t)^2 + 2t\sigma' = \varepsilon[1 - (1-\varepsilon)t] \quad \Longleftrightarrow \quad \sigma' = \frac{\varepsilon[1 - (1-\varepsilon)t] - (1-t)^2}{2t}$$

and let

$$\sigma^* = \max\{\sigma', 0\}.$$

The quantity σ^* equals 0 if and only if

$$\varepsilon[1 - (1-\varepsilon)t] \leq (1-t)^2$$

and it is easy to verify that this happens whenever

$$t \leq t_1(\varepsilon) := 1 - \frac{\varepsilon - \varepsilon^2 + \varepsilon\sqrt{\varepsilon^2 - 2\varepsilon + 5}}{2}, \quad t \geq t_2(\varepsilon) := 1 + \frac{-\varepsilon + \varepsilon^2 + \varepsilon\sqrt{\varepsilon^2 - 2\varepsilon + 5}}{2}.$$

REMARK 2.20 | For small $\varepsilon > 0$ we have $t_2(\varepsilon) < \frac{1}{1-\varepsilon}$, $0 < t_1(\varepsilon) < 1$

Hence

$$\sigma^* = \max\{\sigma', 0\} = \begin{cases} \frac{\varepsilon[1 - (1-\varepsilon)t] - (1-t)^2}{2t} & \text{if } t \in (t_1(\varepsilon), t_2(\varepsilon)) \subseteq \left(0, \frac{1}{1-\varepsilon}\right) \\ 0 & \text{if } t \in (0, t_1(\varepsilon)) \cup \left(t_2(\varepsilon), \frac{1}{1-\varepsilon}\right). \end{cases}$$

We split now integral (2.28) into four pieces, as following

$$\begin{aligned} \int_0^{\frac{1}{1-\varepsilon}} dt \int_0^1 d\sigma &= \\ &= \int_{t_1(\varepsilon)}^{t_2(\varepsilon)} dt \int_0^{\sigma^*} d\sigma + \int_{t_1(\varepsilon)}^{t_2(\varepsilon)} dt \int_{\sigma^*}^1 d\sigma + \int_0^{t_1(\varepsilon)} dt \int_0^1 d\sigma + \int_{t_2(\varepsilon)}^{\frac{1}{1-\varepsilon}} dt \int_0^1 d\sigma, \end{aligned} \quad (2.29)$$

and we treat each of them separately:

- for the first one we have

$$\begin{aligned} \int_0^{\sigma^*} \frac{\sigma^{(N-3)/2}}{[(1-t)^2 + 2t\sigma]^{N/2-s}} d\sigma &\leq \frac{1}{(2t)^{(N-3)/2}} \int_0^{\sigma^*} [(1-t)^2 + 2t\sigma]^{s-3/2} d\sigma \leq \\ &\leq \frac{1}{t^{(N-1)/2}} \frac{1}{s - \frac{1}{2}} \left[\varepsilon^{s-1/2} (1 - (1-\varepsilon)t)^{s-1/2} - |1-t|^{2s-1} \right] \end{aligned}$$

and therefore the first integral is less than

$$\int_{t_1(\varepsilon)}^{t_2(\varepsilon)} \frac{t^{(N-1)/2}}{(1 - (1-\varepsilon)t)^\beta} \cdot \frac{1}{s - \frac{1}{2}} \left[\varepsilon^{s-1/2} (1 - (1-\varepsilon)t)^{s-1/2} - |1-t|^{2s-1} \right] dt, \quad (2.30)$$

note now that $t_2(\varepsilon) - t_1(\varepsilon) \sim \varepsilon$, up to a multiplicative constant, and $\frac{1}{1-\varepsilon} - 1 \sim \varepsilon$ and thus (2.30) is of magnitude

$$\varepsilon^{-\beta+2s},$$

- for the second integral we have

$$\begin{aligned} \int_{\sigma^*}^1 \frac{\sigma^{(N-3)/2}}{[(1-t)^2 + 2t\sigma]^{N/2}} d\sigma &\leq \frac{1}{(2t)^{(N-3)/2}} \int_0^{\sigma^*} [(1-t)^2 + 2t\sigma]^{-3/2} d\sigma \leq \\ &\leq \frac{1}{t^{(N-1)/2}} \left[\varepsilon^{-1/2} (1 - (1-\varepsilon)t)^{-1/2} - (1+t^2)^{-1/2} \right] \end{aligned}$$

and therefore the second integral is less than

$$\int_{t_1(\varepsilon)}^{t_2(\varepsilon)} \frac{t^{(N-1)/2}}{(1 - (1-\varepsilon)t)^\beta} \cdot \left[\varepsilon^{-1/2} (1 - (1-\varepsilon)t)^{-1/2} - (1+t^2)^{-1/2} \right] dt; \quad (2.31)$$

and (2.31) is of magnitude

$$\varepsilon^{-\beta+2s};$$

- for the third integral we have

$$\int_0^1 \frac{\sigma^{(N-3)/2}}{[(1-t)^2 + 2t\sigma]^{N/2}} d\sigma \leq \frac{1}{t^{(N-3)/2}} \int_0^1 \frac{d\sigma}{[(1-t)^2 + 2t\sigma]^{3/2}} \leq \frac{1}{t^{(N-1)/2}} \cdot \frac{1}{|1-t|}$$

so that the third integral is less than

$$\begin{aligned} \int_0^{t_1(\varepsilon)} \frac{t^{(N-1)/2} \varepsilon^s}{(1 - (1-\varepsilon)t)^{\beta-s}} \cdot \frac{dt}{1-t} &\leq \varepsilon^s \int_0^{t_1(\varepsilon)} \frac{dt}{(1-t)(1 - (1-\varepsilon)t)^{\beta-s}} \leq \\ &\leq \varepsilon^s \int_0^{t_1(\varepsilon)} \frac{dt}{(1-t)^{\beta-s+1}} = \frac{\varepsilon^s (1 - t_1(\varepsilon))^{\beta-s}}{\beta-s} - \frac{\varepsilon^s}{\beta-s} \end{aligned} \quad (2.32)$$

if $\beta \neq s$ and (2.32) is of magnitude

$$\begin{aligned} \varepsilon^s &\quad \text{for } 0 \leq \beta < s, \\ \varepsilon^s \log \frac{1}{\varepsilon} &\quad \text{for } \beta = s, \\ \varepsilon^{-\beta+2s} &\quad \text{for } s < \beta < 1+s; \end{aligned}$$

- for the fourth integral we have

$$\int_0^1 \frac{\sigma^{(N-3)/2}}{[(1-t)^2 + 2t\sigma]^{N/2}} d\sigma \leq \frac{1}{t^{(N-1)/2}} \cdot \frac{1}{|1-t|}$$

so that the fourth integral is less than

$$\int_{t_2(\varepsilon)}^{\frac{1}{1-\varepsilon}} \frac{t^{(N-1)/2} \varepsilon^s}{(1 - (1-\varepsilon)t)^{\beta-s}} \cdot \frac{dt}{t-1} \leq \varepsilon^s \int_{t_2(\varepsilon)}^{\frac{1}{1-\varepsilon}} \frac{dt}{(t-1)(1 - (1-\varepsilon)t)^{\beta-s}} \quad (2.33)$$

and (2.33) is of magnitude

$$\begin{aligned} \varepsilon^s \int_{t_2(\varepsilon)}^{\frac{1}{1-\varepsilon}} \frac{dt}{(t-1)^{\beta-s+1}} &\sim \varepsilon^s \quad \text{for } 0 \leq \beta < s, \\ \varepsilon^s \int_{t_2(\varepsilon)}^{\frac{1}{1-\varepsilon}} \frac{dt}{t-1} &\sim \varepsilon^s \log \frac{1}{\varepsilon} \quad \text{for } \beta = s, \\ \frac{\varepsilon^s}{t_2(\varepsilon)-1} \int_{t_2(\varepsilon)}^{\frac{1}{1-\varepsilon}} \frac{dt}{(1 - (1-\varepsilon)t)^{\beta-s}} &\sim \varepsilon^{-\beta+2s} \quad \text{for } s < \beta < 1+s. \end{aligned}$$

Resuming the information collected so far, what we have gained is that, up to constants,

$$u(x) \leq \begin{cases} \delta(x)^s & \text{for } 0 \leq \beta < s, \\ \delta(x)^s \log \frac{1}{\delta(x)} & \text{for } \beta = s, \\ \delta(x)^{-\beta+2s} & \text{for } s < \beta < 1+s. \end{cases} \quad (2.34)$$

This establishes an upper bound for the solutions. Note now that the integral (2.28) works also as a lower bound, of course up to constants. Using the split expression (2.29) we entail

$$\begin{aligned} u(x) &\geq \int_0^{\frac{1}{1-\varepsilon}} \int_0^1 \frac{\{[(1-t)^2 + 1 + 2t\sigma] \wedge \varepsilon[1 - (1-\varepsilon)t]\}^s}{[(1-t)^2 + 2t\sigma]^{N/2}} \sigma^{(N-3)/2} d\sigma \frac{t^{N-1}}{(1 - (1-\varepsilon)t)^\beta} dt \\ &\geq \int_0^{t_1(\varepsilon)} \frac{t^{N-1} \varepsilon^s}{(1 - (1-\varepsilon)t)^{\beta-s}} \int_0^1 \frac{\sigma^{(N-3)/2}}{[(1-t)^2 + 2t\sigma]^{N/2}} d\sigma dt, \end{aligned}$$

where we have used only the expression with the third integral in (2.29). We claim that

$$\int_0^1 \frac{\sigma^{(N-3)/2}}{[(1-t)^2 + 2t\sigma]^{N/2}} d\sigma \geq \frac{1}{1-t} \geq \frac{1}{1 - (1-\varepsilon)t}, \quad (2.35)$$

where the inequality is intended to hold up to constants. In case (2.35) holds and $\beta \neq s$ we have

$$\begin{aligned} \int_0^{t_1(\varepsilon)} \frac{t^{N-1} \varepsilon^s}{(1 - (1-\varepsilon)t)^{\beta-s}} \int_0^1 \frac{\sigma^{(N-3)/2}}{[(1-t)^2 + 2t\sigma]^{N/2}} d\sigma dt &\geq \int_0^{t_1(\varepsilon)} \frac{t^{N-1} \varepsilon^s}{(1 - (1-\varepsilon)t)^{\beta-s+1}} dt = \\ &= \frac{1}{(1-\varepsilon)(\beta-s)} \cdot \frac{t^{N-1} \varepsilon^s}{(1 - (1-\varepsilon)t)^{\beta-s}} \Big|_0^{t_1(\varepsilon)} - \frac{N-1}{(1-\varepsilon)(\beta-s)} \int_0^{t_1(\varepsilon)} \frac{t^{N-2} \varepsilon^s}{(1 - (1-\varepsilon)t)^{\beta-s}} dt. \end{aligned}$$

The integral on the second line is a bounded quantity as $\varepsilon \downarrow 0$ since $\beta < 1+s$. Now, we are left with

$$u(x) \geq \frac{\varepsilon^{-\beta+2s}}{\beta-s} - \frac{\varepsilon^s}{\beta-s}$$

so that

$$u(x) \geq \begin{cases} \varepsilon^s & \text{when } 0 \leq \beta < s \\ \varepsilon^s \log \frac{1}{\varepsilon} & \text{when } \beta = s \\ \varepsilon^{-\beta+2s} & \text{when } s < \beta < 1+s. \end{cases}$$

We still have to prove (2.35): note that an integration by parts yields, for $N \geq 4$,

$$\begin{aligned} \int_0^1 \frac{\sigma^{(N-3)/2}}{[(1-t)^2 + 2t\sigma]^{N/2}} d\sigma &= \frac{1}{2t(1-N/2)} \cdot \frac{1}{[(1-t)^2 + 2t]^{N/2-1}} + \\ &\quad + \frac{1}{2t(N/2-1)} \int_0^1 \frac{\sigma^{(N-5)/2}}{[(1-t)^2 + 2t\sigma]^{N/2-1}} d\sigma \end{aligned}$$

so that we can show (2.35) only in dimensions $N = 2, 3$ and deduce the same conclusions for any other value of N by integrating by parts a suitable number of times. For $N = 2$

$$\int_0^1 \frac{d\sigma}{\sqrt{\sigma}[(1-t)^2 + 2t\sigma]} = \frac{\pi}{\sqrt{2t}(1-t)}.$$

For $N = 3$

$$\int_0^1 \frac{d\sigma}{[(1-t)^2 + 2t\sigma]^{3/2}} = \frac{1}{4t} \left(\frac{1}{1-t} - \sqrt{\frac{1}{1+t^2}} \right)$$

which completely proves our claim (2.35).

So far we have worked only on spherical domains. In a general domain Ω with $C^{1,1}$ boundary, split the problem in two by setting $u = u_1 + u_2$, for $0 < \beta < 1 + s$ and a small $\delta_0 > 0$

$$\begin{cases} (-\Delta)^s u_1 = \frac{\chi_{\{\delta \geq \delta_0\}}}{\delta^\beta} & \text{in } \Omega \\ u_1 = 0 & \text{in } \mathcal{C}\Omega \\ Eu_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} (-\Delta)^s u_2 = \frac{\chi_{\{\delta < \delta_0\}}}{\delta^\beta} & \text{in } \Omega \\ u_2 = 0 & \text{in } \mathcal{C}\Omega \\ Eu_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that $u_1 \in C^s(\mathbb{R}^N)$, see [73, Proposition 1.1]. Since $\partial\Omega \in C^{1,1}$, we choose now δ_0 sufficiently small in order to have that for any $y \in \Omega$ with $\delta(y) < \delta_0$ it is uniquely determined $\theta = \theta(y) \in \partial\Omega$ such that $|y - \theta| = \delta(y)$. Then in inequalities (see (1.14))

$$\begin{aligned} \int_{\{\delta(y) < \delta_0\}} \frac{[|x - y|^2 \wedge \delta(x)\delta(y)]^s}{c_2 |x - y|^N \delta(y)^\beta} dy &\leq u_2(x) = \int_{\{\delta(y) < \delta_0\}} \frac{G_\Omega(x, y)}{\delta(y)^\beta} dy \leq \\ &\leq \int_{\{\delta(y) < \delta_0\}} \frac{c_2 [|x - y|^2 \wedge \delta(x)\delta(y)]^s}{|x - y|^N \delta(y)^\beta} dy \end{aligned}$$

set

$$\begin{aligned} \nu(\theta), \theta \in \partial\Omega, \text{ the outward unit normal to } \partial\Omega \text{ at } \theta, \\ \varepsilon := \delta(x), \theta^* = \theta(x), \text{ and } x = \theta^* - \varepsilon \nu(\theta^*), \\ \theta = \theta(y), r := \delta(y), \text{ and } y = \theta - r \nu(\theta). \end{aligned}$$

Then, using the Fubini's Theorem, we write

$$\int_{\{\delta(y) < \delta_0\}} \frac{[|x - y|^2 \wedge \delta(x)\delta(y)]^s}{|x - y|^N \delta(y)^\beta} dy = \int_0^{\delta_0} \int_{\partial\Omega} \frac{[|\theta^* - \varepsilon \nu(\theta^*) - \theta + r \nu(\theta)|^2 \wedge \varepsilon r]^s}{|\theta^* - \varepsilon \nu(\theta^*) - \theta + r \nu(\theta)|^N r^\beta} d\mathcal{H}(\theta) dr. \quad (2.36)$$

Split the integration on $\partial\Omega$ into the integration on $\Gamma := \{\theta \in \partial\Omega : |\theta - \theta^*| < \delta_1\}$ and $\partial\Omega \setminus \Gamma$, and choose $\delta_1 > 0$ small enough to have a $C^{1,1}$ diffeomorphism

$$\begin{aligned} \varphi : \tilde{\Gamma} \subseteq S^{N-1} &\longrightarrow \Gamma \\ \omega &\longmapsto \theta = \varphi(\omega). \end{aligned}$$

We build now

$$\begin{aligned} \bar{\varphi} : \tilde{\Gamma} \times (0, \delta_0) &\subseteq B_1 \longrightarrow \{y \in \Omega : \delta(y) < \delta_0, |\theta(y) - \theta^*| < \delta_1\} \\ (1 - \delta)\omega &= \omega - \delta\omega \longmapsto y = \theta - \delta \nu(\theta) = \varphi(\omega) - \delta \nu(\varphi(\omega)), \end{aligned}$$

where we suppose $e_1 \in \tilde{\Gamma}$ and $\phi(e_1) = \theta^*$. With this change of variables (2.36) becomes

$$\int_{\{\delta(y) < \delta_0\}} \frac{[|x - y|^2 \wedge \delta(x)\delta(y)]^s}{|x - y|^N \delta(y)^\beta} dy = \int_0^{\delta_0} \int_{\tilde{\Gamma}} \frac{[|e_1 - \varepsilon e_1 - \omega + r \omega|^2 \wedge \varepsilon r]^s}{|e_1 - \varepsilon e_1 - \omega + r \omega|^N r^\beta} |D\bar{\varphi}(\omega)| d\mathcal{H}(\omega) dr$$

which, since $|D\bar{\varphi}(\omega)|$ is a bounded continuous quantity far from 0, is bounded below and above in terms of

$$\int_0^{\delta_0} \int_{\tilde{\Gamma}} \frac{[|(1 - \varepsilon)e_1 - (1 - r)\omega|^2 \wedge \varepsilon r]^s}{|(1 - \varepsilon)e_1 - (1 - r)\omega|^N r^\beta} d\mathcal{H}(\omega) dr,$$

i.e. we are brought back to the spherical case.

2.4.2 Boundary continuity of s -harmonic functions

Consider $g : \mathcal{C}\Omega \rightarrow \mathbb{R}$ and

$$u(x) = - \int_{\mathcal{C}\Omega} g(y) (-\Delta)^s G_\Omega(x, y) dy$$

and think of letting $x \rightarrow \theta \in \partial\Omega$. Suppose that for any small $\varepsilon > 0$ there exists $\delta > 0$ such that $|g(y) - g(\theta)| < \varepsilon$ for any $y \in \mathcal{C}\Omega \cap B_\delta(\theta)$. Then

$$\begin{aligned} |u(x) - g(\theta)| &= \left| \int_{\mathcal{C}\Omega} (g(\theta) - g(y))(-\Delta)^s G_\Omega(x, y) dy \right| \leq \\ &\leq \int_{\mathcal{C}\Omega \cap B_\delta(\theta)} |(g(\theta) - g(y))(-\Delta)^s G_\Omega(x, y)| dy + \int_{\mathcal{C}\Omega \setminus B_\delta(\theta)} |(g(\theta) - g(y))(-\Delta)^s G_\Omega(x, y)| dy. \end{aligned}$$

The first addend satisfies

$$\int_{\mathcal{C}\Omega \cap B_\delta(\theta)} |(g(\theta) - g(y))(-\Delta)^s G_\Omega(x, y)| dy \leq -\varepsilon \int_{\mathcal{C}\Omega \cap B_\delta(\theta)} (-\Delta)^s G_\Omega(x, y) dy \leq \varepsilon.$$

For the second one we exploit (1.13):

$$\int_{\mathcal{C}\Omega \setminus B_\delta(\theta)} |(g(\theta) - g(y))(-\Delta)^s G_\Omega(x, y)| dy \leq \delta(x)^s \int_{\mathcal{C}\Omega \setminus B_\delta(\theta)} \frac{c_1 |g(\theta) - g(y)|}{\delta(y)^s (1 + \delta(y))^s |x - y|^N} dy$$

which converges to 0 as $x \rightarrow \theta \in \partial\Omega$. So we have that

$$\lim_{x \rightarrow \theta} |u(x) - g(\theta)| \leq \varepsilon,$$

and by arbitrarily choosing ε , we conclude

$$\lim_{x \rightarrow \theta} u(x) = g(\theta).$$

2.4.3 Explosion rate of large s -harmonic functions: proof of (1.17)

We study here the rate of divergence of

$$u(x) = - \int_{\mathcal{C}\Omega} g(y) (-\Delta)^s G_\Omega(x, y) dy \quad \text{as } x \rightarrow \partial\Omega, \ x \in \Omega$$

which is the solution to

$$\begin{cases} (-\Delta)^s u = 0 & \text{in } \Omega \\ u = g & \text{in } \mathcal{C}\Omega \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

in case g explodes at $\partial\Omega$.

REMARK 2.21 | The asymptotic behaviour of u depends only on the values of g near the boundary, since we can split

$$g = g\chi_{\{d < \eta\}} + g\chi_{\{d \geq \eta\}}$$

and the second addend has a null contribution on the boundary, in view of Paragraph 2.4.2. Therefore in our computations we will suppose that $g(y) = 0$ for $\delta(y) > \eta$.

In the further assumption that g explodes like a power, i.e. there exist $\eta, k, K > 0$ for which

$$\frac{k}{\delta(y)^\tau} \leq g(y) \leq \frac{K}{\delta(y)^\sigma}, \quad \text{for } 0 \leq \tau \leq \sigma < 1 - s, \ 0 < \delta(y) < \eta$$

(the choice $\sigma < 1 - s$ is in order to have (2.18), see (1.13) above) our proof doesn't require heavy computations and it is as follows.

Dropping multiplicative constants in inequalities and for $\Omega_\eta = \{y \in \mathcal{C}\Omega : \delta(y) < \eta\}$:

$$\begin{aligned} \delta(x)^\sigma u(x) &= -\delta(x)^\sigma \int_{\mathcal{C}\Omega} g(y) \cdot (-\Delta)^s G_\Omega(x, y) dy \leq \\ &\leq \int_{\Omega_\eta} \frac{\delta(x)^{s+\sigma}}{\delta(y)^{s+\sigma} (1 + \delta(y))^s |x - y|^N} dy \leq - \int_{\mathcal{C}\Omega} \chi_{\Omega_\eta}(y) \cdot (-\Delta)^{s+\sigma} G_\Omega^{s+\sigma}(x, y) dy \leq 1 \end{aligned}$$

Similarly one can treat also the lower bound:

$$\begin{aligned} \delta(x)^\tau u(x) &= -\delta(x)^\tau \int_{\mathcal{C}\Omega} g(y) \cdot (-\Delta)^s G_\Omega(x, y) dy \geq \\ &\geq \int_{\Omega_\eta} \frac{\delta(x)^{s+\tau}}{\delta(y)^{s+\tau} (1+\delta(y))^s |x-y|^N} dy \geq - \int_{\mathcal{C}\Omega} \chi_{\Omega_\eta}(y) \cdot (-\Delta)^{s+\tau} G_\Omega^{s+\tau}(x, y) dy \xrightarrow{x \rightarrow \partial\Omega} 1. \end{aligned}$$

The limit we have computed above is the continuity up to the boundary of \widehat{u} solution of

$$\begin{cases} (-\Delta)^{s+\tau} \widehat{u} = 0 & \text{in } \Omega, \\ \widehat{u} = \chi_{\Omega_\eta} & \text{in } \mathcal{C}\Omega, \\ E\widehat{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

REMARK 2.22 | Both the upper and the lower estimate are optimal, thanks to what have been shown in Example 2.11.

In the case of a general boundary datum g we start from the case $\Omega = B$, recalling that in this setting, according to [60, equation (1.6.11')],

$$-(-\Delta)^s G_B(x, y) = \frac{c(N, s)}{|x-y|^N} \left(\frac{1-|x|^2}{|y|^2-1} \right)^s$$

and therefore

$$u(x) = \int_{CB} \frac{c(N, s)}{|x-y|^N} \left(\frac{1-|x|^2}{|y|^2-1} \right)^s g(y) dy.$$

Suppose without loss of generality

$$\begin{aligned} x &= (1-\varepsilon) e_1 & \varepsilon &= \delta(x) \\ y &= (1+r) \theta & r &= \delta(y), \quad \theta = \frac{y}{|y|}, \quad \theta_1 = e_1 \cdot \theta \end{aligned}$$

so that

$$u(x) = \int_0^\eta \left[\int_{\partial B} \frac{c(N, s)}{|(1-\varepsilon)^2 + (1+r)^2 - 2(1-\varepsilon)(1+r)\theta_1|^{N/2}} \left(\frac{\varepsilon(2-\varepsilon)}{r(2+r)} \right)^s g(r\theta) d\mathcal{H}(\theta) \right] (1+r)^{N-1} dr.$$

Denote now by $\overline{g}(r) = \sup_{r \leq t \leq \eta} \sup_{\delta(x)=t} g(x)$, which is decreasing in 0. Splitting the integral in the θ variable into two integrals in the variables (θ_1, θ') where $\theta_1^2 + |\theta'|^2 = |\theta|^2 = 1$, up to constants we obtain

$$\begin{aligned} u(x) &\leq \int_0^\eta \left[\int_{-1}^1 \frac{(1-\theta_1^2)^{(N-3)/2}}{|(1-\varepsilon)^2 + (1+r)^2 - 2(1-\varepsilon)(1+r)\theta_1|^{N/2}} \cdot \frac{\varepsilon^s}{r^s} d\theta_1 \right] \overline{g}(r) (1+r)^{N-1} dr \leq \\ &\leq \varepsilon^s \int_0^\eta \left[\int_{-1}^1 \frac{(1-\theta_1^2)^{(N-3)/2}}{|(1-\varepsilon)^2 + (1+r)^2 - 2(1-\varepsilon)(1+r)\theta_1|^{N/2}} d\theta_1 \right] \frac{\overline{g}(r)}{r^s} dr. \quad (2.37) \end{aligned}$$

Define $M := \frac{1+r}{1-\varepsilon} > 1$ and look at the inner integral:

$$\begin{aligned} \int_{-1}^1 \frac{(1-\theta_1^2)^{(N-3)/2}}{|1+M^2-2M\theta_1|^{N/2}} d\theta_1 &= \int_{-1}^1 \frac{(1-\theta_1^2)^{(N-3)/2}}{|1-\theta_1^2+(M-\theta_1)^2|^{N/2}} d\theta_1 \leq \\ &\leq \int_{-1}^1 |1-\theta_1^2+(M-\theta_1)^2|^{-3/2} d\theta_1. \end{aligned}$$

The integral from -1 to 0 contributes by a bounded quantity so that we are left with

$$\begin{aligned} u(x) &\leq \int_0^1 |1-\theta_1+(M-\theta_1)^2|^{-3/2} d\theta_1 = \int_0^1 |\tau+(M-1+\tau)^2|^{-3/2} d\tau \\ &\leq \int_0^1 |\tau+(M-1)^2|^{-3/2} d\tau = -2 \left(\tau+(M-1)^2 \right)^{-1/2} \Big|_{\tau=0}^1 \leq \frac{1}{M-1} = \frac{1-\varepsilon}{r+\varepsilon}. \end{aligned}$$

Thus

$$u(x) \leq \varepsilon^s \int_0^\eta \frac{\bar{g}(r)}{r^s} \cdot \frac{1-\varepsilon}{r+\varepsilon} dr \leq \varepsilon^s \int_0^\varepsilon \frac{\bar{g}(r)}{r^s} \cdot \frac{1}{r+\varepsilon} dr + \int_1^{\eta/\varepsilon} \frac{\bar{g}(\varepsilon\tau)}{\tau^s} \cdot \frac{1}{1+\tau} d\tau.$$

Our claim now is that this last expression is controlled by $\bar{g}(\varepsilon)$ as $\varepsilon \downarrow 0$. Since \bar{g} is exploding in 0, for small ε it holds $\bar{g}(\varepsilon\tau) \leq \bar{g}(\varepsilon)$ for $\tau > 1$ and

$$\frac{1}{\bar{g}(\varepsilon)} \int_1^{\eta/\varepsilon} \frac{\bar{g}(\varepsilon\tau)}{\tau^s} \cdot \frac{1}{1+\tau} d\tau \leq \int_1^{+\infty} \frac{1}{\tau^s} \cdot \frac{1}{1+\tau} d\tau.$$

For the other integral

$$\frac{\varepsilon^s}{\bar{g}(\varepsilon)} \int_0^\varepsilon \frac{\bar{g}(r)}{r^s} \cdot \frac{1}{r+\varepsilon} dr \leq \frac{\varepsilon^s}{\varepsilon \bar{g}(\varepsilon)} \int_0^\varepsilon \frac{\bar{g}(r)}{r^s} dr.$$

To compute the limit as $\varepsilon \downarrow 0$ we use a Taylor expansion:

$$\frac{1}{\varepsilon} \int_0^\varepsilon G(r) dr = G(\varepsilon) + G'(\varepsilon) \frac{\varepsilon^2}{2} + o(\varepsilon^2),$$

where we have denoted by $G(\varepsilon) = \varepsilon^{-s} \bar{g}(\varepsilon)$. Thus

$$\frac{\varepsilon^s}{\bar{g}(\varepsilon)} \int_0^\varepsilon \frac{\bar{g}(r)}{r^s} \cdot \frac{1}{r+\varepsilon} dr \leq 1 + \frac{G'(\varepsilon)}{2G(\varepsilon)} \varepsilon + o\left(\frac{\varepsilon}{G(\varepsilon)}\right).$$

We are going to show now that

$$\frac{|G'(\varepsilon)|}{G(\varepsilon)} \varepsilon < 1.$$

Indeed

$$-\int_\varepsilon^\eta \frac{G'(\xi)}{G(\xi)} d\xi < \int_\varepsilon^\eta \frac{d\xi}{\xi} \iff G(\varepsilon) < G(\eta) \frac{\eta}{\varepsilon}$$

which is guaranteed by the fact that G is integrable in a neighbourhood of 0.

These computations show that, in the case of the ball, the explosion rate of the s -harmonic function induced by a large boundary datum is the almost the same as the rate of the datum itself.

Note now that up to (2.37) the same computations provide a lower estimate for u if we substitute \bar{g} with $\underline{g}(r) = \inf_{\delta(x)=r} g(x)$. Then

$$\begin{aligned} \int_{-1}^1 \frac{(1-\theta_1^2)^{(N-3)/2}}{|1+M^2-2M\theta_1|^{N/2}} d\theta_1 &= \int_{-1}^1 \frac{(1-\theta_1^2)^{(N-3)/2}}{|1-\theta_1^2+(M-\theta_1)^2|^{N/2}} d\theta_1 \geq \\ &\geq \int_0^1 \frac{\sigma^{(N-3)/2}}{|\sigma+(M-1+\sigma)^2|^{N/2}} d\sigma \geq \frac{1}{M-1} \end{aligned}$$

where the last inequality is (2.35). Finally we need only to repeat the above computations replacing \bar{g} with \underline{g} and other minor modifications.

In the case of a general smooth domain, we can reduce to the spherical case as we did to conclude Paragraph 2.4.1.

Semilinear fractional Dirichlet problems

3.1 THE METHOD OF SUB- AND SUPERSOLUTION: PROOF OF THEOREM 1.11

The proof is an adaptation of the result by Clément and Sweers [28].

Existence. We can reduce the problem to homogeneous boundary condition, indeed by considering the solution of

$$\begin{cases} (-\Delta)^s v = 0 & \text{in } \Omega \\ v = g & \text{in } \mathcal{C}\Omega \\ Ev = 0 & \text{on } \partial\Omega \end{cases}$$

we can think of solving the problem (note that the boundedness of g implies (1.11))

$$\begin{cases} (-\Delta)^s u = -f(x, v + u) & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

therefore from now on we will suppose $g \equiv 0$. Note also that since v is continuous in Ω and bounded then $(x, t) \mapsto f(x, v(x) + t)$ satisfies *f.1* too.

Modify f by defining

$$F(x, u) = \begin{cases} f(x, \bar{u}(x)) & \text{if } u > \bar{u}(x) \\ f(x, u) & \text{if } \underline{u}(x) \leq u \leq \bar{u}(x) \\ f(x, \underline{u}(x)) & \text{if } u < \underline{u}(x) \end{cases} \quad \text{for every } x \in \bar{\Omega}, u \in \mathbb{R} :$$

the function $F(x, u)$ is continuous and bounded on $\Omega \times \mathbb{R}$, by hypothesis *f.1*) and the boundedness of \underline{u}, \bar{u} . We can write a solution of

$$(*) \quad \begin{cases} (-\Delta)^s u = -F(x, u) & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

as a fixed-point of the map obtained as the composition

$$\begin{array}{ccccc} L^\infty(\Omega) & \longrightarrow & L^\infty(\Omega) & \longrightarrow & L^\infty(\Omega) \\ u & \longmapsto & -F(x, u(x)) & \longmapsto & w \text{ s.t. } \begin{cases} (-\Delta)^s w = -F(x, u(x)) \text{ in } \Omega, \\ w = 0 \text{ in } \mathcal{C}\Omega, \quad Ew = 0 \text{ on } \partial\Omega. \end{cases} \end{array}$$

The first map sends $L^\infty(\Omega)$ in a bounded subset of $L^\infty(\Omega)$, by continuity of f and boundedness of \underline{u}, \bar{u} . The second map is compact since $w \in C^s(\mathbb{R}^N)$, thanks to the results in¹ [73, Proposition 1.1]. Then the composition admits a fixed point in view of the Schauder Fixed Point Theorem.

¹The notion of solution used in [73] is different from the one we use here, but look at Section 5.2.

Note that a solution to the original problem lying between \underline{u} and \bar{u} in Ω , is also a solution of (*). Moreover, *any* solution of (*) is between \underline{u} and \bar{u} . Indeed consider $A := \{x \in \Omega : u(x) > \bar{u}(x)\}$, which is open by the continuity of u and \bar{u} . For any $\psi \in C_c^\infty(A)$, $\psi \geq 0$, with the corresponding $\phi_A \in \mathcal{T}(A)$, by Corollary 2.15,

$$\begin{aligned} \int_A u(x) \psi(x) dx &= - \int_A F(x, u(x)) \phi_A(x) dx - \int_{\Omega \setminus A} u(x) (-\Delta)^s \phi_A(x) dx \\ &= - \int_A F(x, \bar{u}(x)) \phi_A(x) dx - \int_{\Omega \setminus A} u(x) (-\Delta)^s \phi_A(x) dx \\ &\leq \int_A \bar{u}(x) \psi(x) dx \end{aligned}$$

which implies $u \leq \bar{u}$ in A , by positivity of ψ , proving $A = \emptyset$.

Uniqueness. If we have two continuous solutions u and w

$$\begin{cases} (-\Delta)^s u = -f(x, u) & \text{in } \Omega \\ u = g & \text{in } \mathcal{C}\Omega \\ Eu = 0 & \text{in } \partial\Omega \end{cases} \quad \begin{cases} (-\Delta)^s w = -f(x, w) & \text{in } \Omega \\ w = g & \text{in } \mathcal{C}\Omega \\ Ew = 0 & \text{in } \partial\Omega \end{cases}$$

then for the difference $u - w$ it holds

$$\begin{cases} (-\Delta)^s (u - w) = -f(x, u) + f(x, w) & \text{in } \Omega \\ u - w = 0 & \text{in } \mathcal{C}\Omega \\ Eu - Ew = 0 & \text{in } \partial\Omega. \end{cases}$$

Defining $\Omega_1 = \{x \in \Omega : w(x) < u(x)\}$, thanks to the monotony of f ,

$$\begin{cases} (-\Delta)^s (u - w) \leq 0 & \text{in } \Omega_1 \\ u - w \leq 0 & \text{in } \mathcal{C}\Omega_1 \\ Eu - Ew = 0 & \text{in } \partial\Omega_1 \end{cases}$$

but then, according to Lemma 2.16, $u \leq w$ in Ω_1 . This means Ω_1 is empty. By reversing the roles of u and w , we deduce $u = w$ in Ω .

Minimal solution. We refer the reader to the proof in [40, Corollary 2.2].

3.2 PROOF OF THEOREM 1.12

In the case of negative right-hand side, Theorem 1.12 follows from Theorem 1.13. So, assume the right-hand side is positive and consider

$$\begin{cases} (-\Delta)^s v = f(x, v) & \text{in } \Omega, \\ \lim_{\substack{x \rightarrow \partial\Omega \\ x \in \Omega}} v(x) = +\infty. \end{cases}$$

We look for a suitable shape g of v outside Ω and exploding at $\partial\Omega$: the large s -harmonic function v_0 induced by g in Ω will be a subsolution of our equation, and in particular will imply that the blow-up condition at $\partial\Omega$ is fulfilled. Then, in order to prove the existence part, we need a supersolution.

Consider $F : \mathbb{R} \rightarrow \mathbb{R}$ continuous, increasing and such that $F(t) \geq f(x, t)$ for any $t \geq 0$: for example,

$$F(t) = t^{\frac{4s}{1-s}} + \max_{0 \leq r \leq t} \left[\max_{x \in \bar{\Omega}} f(x, r) \right].$$

Choose

$$g(x) := \frac{F^{-1}(I(x)) - 1}{\bar{c}},$$

where $\bar{c} = \bar{c}(N, s, \Omega)$ is the constant of equation (1.17) giving the upper control of large s -harmonic functions in terms of the boundary datum (see Paragraph 2.4.3), and define for $x \in \mathbb{R}^N$, $\delta(x) \leq \max_{x' \in \bar{\Omega}} \delta(x')$

$$I(x) = \min \left\{ C_{N,s} \int_{\mathcal{C}\Omega} \frac{dy}{|z-y|^{N+2s}} : z \in \Omega, \delta(z) = \delta(x) \right\},$$

while $I(x) = 0$ when $x \in \mathcal{C}\Omega$, $\delta(x) > \max_{x' \in \bar{\Omega}} \delta(x')$. Note that when $\delta(x)$ is small

$$I(x) \leq C_{N,s} \omega_{N-1} \int_{\delta(x)}^{+\infty} \frac{d\rho}{\rho^{1+2s}} = \frac{C_{N,s} \omega_{N-1}}{2s} \cdot \frac{1}{\delta(x)^{2s}}.$$

Such g satisfies hypothesis (2.18), since when $\delta(x)$ is small

$$F(t) \geq t^{\frac{4s}{1-s}} \Rightarrow F^{-1}(t) \leq t^{\frac{1-s}{4s}} \Rightarrow F^{-1}(I(x)) \leq \left(\frac{C_{N,s} \omega_{N-1}}{2s} \right)^{\frac{1-s}{4s}} \frac{1}{\delta(x)^{(1-s)/2}}.$$

Call v_0 the solution to

$$\begin{cases} (-\Delta)^s v_0 = 0 & \text{in } \Omega, \\ v_0 = g & \text{in } \mathcal{C}\Omega, \\ E v_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Denote by $w := v - v_0$: our claim is that problem

$$\begin{cases} (-\Delta)^s w = f(x, v_0 + w) & \text{in } \Omega \\ w = 0 & \text{in } \mathcal{C}\Omega \\ E w = 0 & \text{on } \partial\Omega \end{cases}$$

admits a solution w . Indeed, we have a subsolution which is the function constant to 0 in Ω and χ_Ω turns out to be a supersolution. To show this we consider the problem

$$\begin{cases} (-\Delta)^s w \geq F(v_0 + w) & \text{in } \Omega \\ w = 0 & \text{in } \mathcal{C}\Omega \\ E w = 0 & \text{on } \partial\Omega \end{cases}$$

and observe that, when $x \in \Omega$

$$(-\Delta)^s \chi_\Omega(x) = C_{N,s} \int_{\mathcal{C}\Omega} \frac{dy}{|x-y|^{N+2s}} \geq I(x)$$

and

$$F(v_0(x) + 1) \leq F(\bar{c}g(x) + 1) = F(F^{-1}(I(x))) = I(x).$$

Finally, the property $F(v_0 + w) \geq f(x, v_0 + w)$ concludes the construction of the supersolution. Then Lemma 3.2 below provides the existence of a solution.

3.3 PROOF OF THEOREM 1.13

For any $n \in \mathbb{N}$, denote by $g_n = \min\{g, n\}$. Also, with the notation of equation (2.14), for a small parameter $r > 0$ denote by

$$f_r(y) = f_r(\rho, \theta) = h(\theta) \frac{\varphi(\rho/r)}{K_r}, \quad K_r = \frac{1}{1+s} \int_{\Omega_r} \varphi(\delta(y)/r) \delta(y)^s dy,$$

and recall that this is an approximation of the h boundary datum. Finally call $u_{n,r}$ the minimal solution of

$$\begin{cases} (-\Delta)^s u_{n,r}(x) = -f(x, u_{n,r}(x)) + f_r(x) & \text{in } \Omega \\ u_{n,r} = g_n & \text{in } \mathcal{C}\Omega \\ Eu_{n,r} = 0 & \text{on } \partial\Omega \end{cases}$$

provided by Theorem 1.11, with $\underline{u} \equiv 0$ and $\bar{u} = u_r^0$ defined below. Note that for any $r > 0$, the sequence $\{u_{n,r}\}_{n \in \mathbb{N}}$ we obtain is increasing in n : indeed, $u_{n+1,r}$ is a supersolution for the problem defining $u_{n,r}$, since it has larger boundary values and the minimality property on $u_{n,r}$ gives $u_{n,r} \leq u_{n+1,r}$. Moreover, $\{u_{n,r}\}_{n \in \mathbb{N}}$ is bounded by the function u_r^0 associated with the linear problem with data g and f_r , i.e.

$$\begin{cases} (-\Delta)^s u_r^0 = f_r & \text{in } \Omega, \\ u_r^0 = g & \text{in } \mathcal{C}\Omega, \\ Eu_r^0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore $u_{n,r}$ admits a pointwise limit in \mathbb{R}^N as $n \uparrow +\infty$. Call u_r this limit: obviously $u_r = g$ in $\mathcal{C}\Omega$. Take any nonnegative $\phi \in \mathcal{T}(\Omega)$ with $0 \leq \psi = (-\Delta)^s \phi|_\Omega \in C_c^\infty(\Omega)$: then

$$\begin{aligned} \int_\Omega [f(x, u_r) - f_r] \phi &\leq \liminf_{n \uparrow +\infty} \int_\Omega [f(x, u_{n,r}) - f_r] \phi = \\ &= -\limsup_{n \uparrow +\infty} \int_\Omega u_{n,r} \psi - \int_{\mathcal{C}\Omega} g (-\Delta)^s \phi = -\int_\Omega u_r \psi - \int_{\mathcal{C}\Omega} g (-\Delta)^s \phi, \end{aligned}$$

where we have used the Fatou lemma and the continuity of the map $t \mapsto f(x, t)$. This means that u_r is a subsolution.

We are left to prove that u_r is also a supersolution. Call $\Omega' = \text{supp} \psi \subset \subset \Omega$ and build a sequence $\{\Omega_k\}_{k \in \mathbb{N}}$ such that $\Omega' \subseteq \Omega_k \subseteq \Omega$ and $\Omega_k \nearrow \Omega$. Since $\psi \in C_c^\infty(\Omega_k)$ for any k , then we can build the sequence of functions $\phi_k \in \mathcal{T}(\Omega_k)$ induced by ψ : this sequence is increasing and converges pointwise to ϕ . Moreover, for any k , since $(-\Delta)^s \phi_k \leq 0$ in $\mathcal{C}\Omega_k$

$$\begin{aligned} \int_\Omega [f(x, u_r) - f_r] \phi_k &= \lim_{n \uparrow +\infty} \int_{\Omega_k} [f(x, u_{n,r}) - f_r] \phi_k \\ &= \lim_{n \uparrow +\infty} \left(-\int_{\Omega_k} u_{n,r} \psi - \int_{\mathcal{C}\Omega_k} u_{n,r} (-\Delta)^s \phi_k \right) \\ &\geq \lim_{n \uparrow +\infty} \left(-\int_\Omega u_{n,r} \psi - \int_{\mathcal{C}\Omega} g_n (-\Delta)^s \phi_k \right) = -\int_\Omega u_r \psi - \int_{\mathcal{C}\Omega} g (-\Delta)^s \phi_k : \end{aligned}$$

letting both sides of the inequality pass to the limit as $k \uparrow +\infty$ we obtain

$$\int_\Omega [-f(x, u_r) - f_r] \phi \geq -\int_\Omega u_r \psi - \int_{\mathcal{C}\Omega} g (-\Delta)^s \phi,$$

because recall that for $x \in \mathcal{C}\Omega \subseteq \mathcal{C}\Omega_k$

$$-(-\Delta)^s \phi_k(x) = \int_{\Omega_k} \frac{\phi_k(y)}{|x - y|^{N+2s}} dy$$

increases to $-(-\Delta)^s \phi(x)$.

This means that u_r is both a sub- and a supersolution and it solves

$$\begin{cases} (-\Delta)^s u_r(x) = -f(x, u_r(x)) + f_r(x) & \text{in } \Omega, \\ u = g & \text{in } \mathcal{C}\Omega, \\ Eu = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

REMARK 3.1 | Note that we have just solved all problems with null h boundary datum, i.e.

$$\begin{cases} (-\Delta)^s u(x) = -f(x, u(x)) & \text{in } \Omega \\ u = g & \text{in } \mathcal{C}\Omega \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

since in this case $f_r \equiv 0$.

We want now to push $r \downarrow 0$. We claim that the family $\{u_r\}_r$ is uniformly bounded and equicontinuous on every compact subset of Ω : there would exist then a $u \in L^1(\Omega)$ and a sequence $\{r_k\}_{k \in \mathbb{N}}$ such that $r_k \rightarrow 0$ as $k \rightarrow +\infty$ and $u_{r_k} \rightarrow u$ a.e. in Ω and uniformly on compact subsets. Then for any $\phi \in \mathcal{T}(\Omega)$

$$\begin{aligned} \int_{\Omega} u_{r_k} (-\Delta)^s \phi &= - \int_{\Omega} f(x, u_{r_k}) \phi + \int_{\Omega} f_{r_k} \phi - \int_{\mathcal{C}\Omega} g (-\Delta)^s \phi \\ &\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \\ \int_{\Omega} u (-\Delta)^s \phi &= - \int_{\Omega} f(x, u) \phi + \int_{\partial\Omega} h D_s \phi - \int_{\mathcal{C}\Omega} g (-\Delta)^s \phi \end{aligned}$$

where the convergence $\int_{\Omega} f(x, u_{r_k}) \phi \rightarrow \int_{\Omega} f(x, u) \phi$ holds since

$$f(x, u_{r_k}) |\phi| \leq \Lambda(a u_{r_k}) C \delta^s \leq \Lambda(a u_0) C \delta^s \leq \Lambda(C \delta^{s-1}) \delta^s \in L^1(\Omega).$$

We still have to prove that our claim is true. The uniform boundedness on compact subsets of $\{u_r\}_r$ is a consequence of inequalities

$$0 \leq u_r \leq u_r^0 \xrightarrow{r \downarrow 0} u_0, \quad \text{for any } r,$$

where

$$\begin{cases} (-\Delta)^s u_r^0 = f_r & \text{in } \Omega \\ u_r^0 = g & \text{in } \mathcal{C}\Omega \\ Eu_r^0 = 0 & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} (-\Delta)^s u_0 = 0 & \text{in } \Omega \\ u_0 = g & \text{in } \mathcal{C}\Omega \\ Eu_0 = h & \text{on } \partial\Omega, \end{cases}$$

and the convergence of v_r to u_0 is uniform in compact subsets of Ω . Then

$$\begin{aligned} |u_r(x) - u_r(z)| &= \left| \int_{\Omega} (G_{\Omega}(x, y) - G_{\Omega}(z, y)) [-f(y, u_r(y)) + f_r(y)] dy \right| \\ &\leq \int_{\Omega} |G_{\Omega}(x, y) - G_{\Omega}(z, y)| \left(a_1 + C a_2 \delta(y)^{p(s-1)} \right) dy + \int_{\Omega} |G_{\Omega}(x, y) - G_{\Omega}(z, y)| C \delta(y)^{-1} dy \end{aligned}$$

implies the equicontinuity. If there exist $b_1, T > 0$ such that

$$f(x, t) \geq b_1 t^{\frac{1+s}{1-s}}, \quad \text{for } t > T.$$

and $h \not\equiv 0$, then a solution would satisfy

$$f(\cdot, u) \delta^s \notin L^1(\Omega);$$

in this case the integration by parts formula defining a weak solution would not make sense.

Note that this proof exploits the negativity of the right-hand side only in considering the s -harmonic function induced by g and h as a supersolution of problem

$$\begin{cases} -(-\Delta)^s u = f(x, u) & \text{in } \Omega \\ u = g & \text{in } \mathcal{C}\Omega \\ Eu = h & \text{in } \partial\Omega. \end{cases}$$

With minor modifications to the proof we can state

LEMMA 3.2 | Let $f : \overline{\Omega} \times \mathbb{R} \rightarrow [0, +\infty)$ be a function satisfying f.1) and f.2). Let $g : \mathcal{C}\Omega \rightarrow \mathbb{R}^+$ be a measurable function satisfying (1.11) and $h \in C(\partial\Omega)$, $h \geq 0$. Assume the nonlinear problem

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega \\ u = g & \text{in } \mathcal{C}\Omega \\ Eu = h & \text{on } \partial\Omega \end{cases}$$

admits a subsolution $\underline{u} \in L^1(\Omega)$ and a supersolution $\overline{u} \in L^1(\Omega)$. Assume also $\underline{u} \leq \overline{u}$ in Ω . Then the above nonlinear problem has a weak solution $u \in L^1(\Omega)$ satisfying

$$\underline{u} \leq u \leq \overline{u}.$$

Proof. Replace, in the above proof, the function u^0 with the supersolution \overline{u} . □

3.4 SUBLINEAR NONNEGATIVE NONLINEARITY: PROOF OF THEOREM 1.15

We first prove a Lemma which will make the proof easily go through.

LEMMA 3.3 | There exists $m = m(\Lambda) > 0$ sufficiently large for which any problem of the form

$$\begin{cases} (-\Delta)^s u(x) = \Lambda(u(x)) & \text{in } \Omega, \\ u = g & \text{in } \mathcal{C}\Omega, \quad g \geq m > 0, \\ Eu = h & \text{on } \partial\Omega, \end{cases}$$

is solvable.

Proof. We can equivalently solve the integral equation

$$u(x) = u_0(x) + \int_{\Omega} G_{\Omega}(x, y) \Lambda(u(y)) dy,$$

where u_0 is the s-harmonic function induced by g and h in Ω .

Define the map

$$\begin{aligned} \mathcal{K} : L^1(\Omega) &\longrightarrow L^1(\Omega) \\ u(x) &\longmapsto u_0(x) + \int_{\Omega} G_{\Omega}(x, y) \Lambda(u(y)) dy \end{aligned}$$

The condition $g \geq m$ in $\mathcal{C}\Omega$ entails $u_0 \geq m$ in Ω ; also, for any $w \in L^1(\Omega)$, $w \geq 0$ implies $\mathcal{K}w \geq u_0 \geq m$, therefore \mathcal{K} sends the subset $D_m := \{w \in L^1(\Omega) : w \geq m\}$ of $L^1(\Omega)$ into itself. Moreover, for $u, v \in D_m$

$$\begin{aligned} \int_{\Omega} |\mathcal{K}u(x) - \mathcal{K}v(x)| dx &\leq \int_{\Omega} |\Lambda(u(y)) - \Lambda(v(y))| \int_{\Omega} G_{\Omega}(x, y) dx dy \leq \\ &\leq \|\zeta\|_{\infty} \sup_{x \in \Omega} \Lambda'(u(x)) \int_{\Omega} |u(y) - v(y)| dy \leq \|\zeta\|_{\infty} \Lambda'(m) \int_{\Omega} |u(y) - v(y)| dy \end{aligned}$$

where $\zeta(y) = \int_{\Omega} G_{\Omega}(x, y) dx$. Now, if m is very large, we have

$$\|\zeta\|_{\infty} \Lambda'(m) < 1,$$

i.e. \mathcal{K} is a contraction on D_m , and \mathcal{K} has a fixed point in D_m . □

In general, for the problem

$$\begin{cases} (-\Delta)^s u = f(x, u) & \text{in } \Omega, \\ u = g & \text{in } \mathcal{C}\Omega, \\ Eu = h & \text{on } \partial\Omega, \end{cases}$$

we have a subsolution which is the s -harmonic function satisfying the boundary conditions.

But we are now able to provide a supersolution: this can be done by setting $g_m = \max\{g, m\}$ and by solving, for some large value of m ,

$$\begin{cases} (-\Delta)^s \bar{u} = \Lambda(\bar{u}) \geq f(x, \bar{u}) & \text{in } \Omega, \\ \bar{u} = g_m \geq g & \text{in } \mathcal{C}\Omega, \\ E\bar{u} = h & \text{on } \partial\Omega. \end{cases}$$

It is sufficient to apply the classical iteration scheme starting from the s -harmonic function u_0 and with iteration step

$$\text{for any } k \in \mathbb{N} \quad \begin{cases} (-\Delta)^s u_k = f(x, u_{k-1}(x)) & \text{in } \Omega, \\ u_k = g & \text{in } \mathcal{C}\Omega, \\ Eu_k = h & \text{on } \partial\Omega. \end{cases}$$

In such a way we build an increasing sequence $\{u_k\}_{k \in \mathbb{N}} \subseteq L^1(\Omega)$ which is uniformly bounded from above by \bar{u} . Indeed, on the one hand we have that

$$\begin{cases} (-\Delta)^s (u_1 - u_0) = f(x, u_0(x)) \geq 0 & \text{in } \Omega \\ u_1 - u_0 = 0 & \text{in } \mathcal{C}\Omega \\ Eu_1 - Eu_0 = 0 & \text{on } \partial\Omega \end{cases}$$

entails that $u_1 - u_0 \geq 0$, while on the other hand an induction argument relying on the monotony of $t \mapsto f(x, t)$ finally shows that u_k increases. Call $u(x) := \lim_k u_k(x)$, which is finite in view of the upper bound furnished by \bar{u} . Then

$$u(x) = \lim_{k \uparrow +\infty} u_k(x) = u_0(x) + \lim_{k \uparrow +\infty} \int_{\Omega} f(y, u_{k-1}(y)) G_{\Omega}(x, y) dy = u_0(x) + \int_{\Omega} f(y, u(y)) G_{\Omega}(x, y) dy.$$

3.5 SUPERLINEAR NONNEGATIVE NONLINEARITY: PROOF OF THEOREM 1.16

We give the proof for problem

$$\begin{cases} (-\Delta)^s u = \lambda f(x, u) & \text{in } \Omega \\ u(x) = \delta(x)^{-\beta} & \text{in } \mathcal{C}\Omega, \quad 0 < \beta < 1 - s \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

while for the other one it is sufficient to replace β with $1 - s$ and repeat the same computations.

To treat the case of a general nonlinearity we use again the equivalent integral equation

$$u(x) = u_0(x) + \lambda \int_{\Omega} G_{\Omega}(x, y) f(y, u(y)) dy,$$

where u_0 is the s -harmonic function induced in Ω by the boundary data. In this case the computations in Section 2.4 on the rate of explosion at the boundary turn out to be very useful. Indeed on the one hand we have that u_0 inherits its explosion from the boundary data g and h : briefly, in our case

$$g(x) = \frac{1}{\delta(x)^{\beta}}, \quad 0 < \beta < 1 - s \quad \longrightarrow \quad \frac{\underline{c}}{\delta(x)^{\beta}} \leq u_0(x) \leq \frac{\bar{c}}{\delta(x)^{\beta}}.$$

Since u_0 is a subsolution, our first goal is to build a supersolution and we build it of the form

$$\bar{u} = u_0 + \zeta,$$

where

$$\begin{cases} (-\Delta)^s \zeta = \delta(x)^{-\gamma} & \text{in } \Omega, \quad \gamma > 0 \\ \zeta = 0 & \text{in } \mathcal{C}\Omega, \\ E\zeta = 0 & \text{on } \partial\Omega. \end{cases}$$

Recall that, (1.16) says

$$\zeta(x) \geq \begin{cases} c_1 \delta(x)^s & \text{if } 0 \leq \gamma < s, \\ c_3 \delta(x)^s \log \frac{1}{\delta(x)} & \text{if } \gamma = s, \\ c_5 \delta(x)^{-\gamma+2s} & \text{if } s < \gamma < 1+s. \end{cases}$$

The function \bar{u} is a supersolution if

$$\bar{u}(x) \geq u_0(x) + \lambda \int_{\Omega} G_{\Omega}(x, y) f(y, \bar{u}(y)) dy,$$

or, equivalently formulated

$$\zeta(x) \geq \lambda \int_{\Omega} G_{\Omega}(x, y) f(y, u_0(y) + \zeta(y)) dy. \quad (3.2)$$

If $f(x, t)$ has an algebraic behavior²

$$f(x, t) \leq a_1 + a_2 t^p, \quad a_1, a_2 > 0, \quad p \geq 1$$

then

$$f(y, u_0(y) + \zeta(y)) \leq a_1 + a_2 (u_0(y) + \zeta(y))^p \leq \begin{cases} \frac{C}{\delta(x)^{p\beta}} & \text{if } \gamma - 2s \leq \beta, \\ \frac{C}{\delta(x)^{p(\gamma-2s)}} & \text{if } \gamma - 2s > \beta. \end{cases}$$

In case $\gamma - 2s \leq \beta$ we have

$$\int_{\Omega} G_{\Omega}(x, y) f(y, u_0(y) + \zeta(y)) dy \leq \begin{cases} c_2 C \delta(x)^s & \text{if } p\beta < s \\ c_4 C \delta(x)^s \log \frac{1}{\delta(x)} & \text{if } p\beta = s \\ \frac{c_6 C}{\delta(x)^{p\beta-2s}} & \text{if } s < p\beta < 1+s \end{cases}$$

again in view of Paragraph 2.4.1, so that we can choose $\gamma = p\beta$ provided $p\beta - 2s \leq \beta$.

If this is not the case then it means we need powers γ satisfying $\gamma - 2s > \beta$. If $\gamma - 2s > \beta$ we have

$$\int_{\Omega} G_{\Omega}(x, y) f(y, u_0(y) + \zeta(y)) dy \leq \begin{cases} c_2 C \delta(x)^s & \text{if } p(\gamma - 2s) < s \\ c_4 C \delta(x)^s \log \frac{1}{\delta(x)} & \text{if } p(\gamma - 2s) = s \\ \frac{c_6 C}{\delta(x)^{p(\gamma-2s)-2s}} & \text{if } s < p(\gamma - 2s) < 1+s \end{cases}$$

and a suitable choice for γ would be

$$\gamma = \max \left\{ \frac{2sp}{p-1}, \beta + 2s + \varepsilon \right\}$$

which fulfils both inequalities

$$\gamma - 2s > \beta, \quad \gamma \geq p(\gamma - 2s).$$

This is an admissible choice for γ provided $\gamma < 1+s$, i.e. only if $p > (1+s)/(1-s)$; in case p doesn't satisfy this lower bound then

$$p \leq \frac{1+s}{1-s} \implies p\beta \leq \frac{1+s}{1-s} \beta \leq \beta + 2s$$

²For $p < 1$ we are actually in the case of the previous paragraph.

and we are in the previous case.

Finally, if $q\beta > 1 + s$ then a solution u should satisfy, whenever $\delta(x) < 1$,

$$f(x, u(x)) \geq bu(x)^q \geq bu_0(x)^q \geq c\delta(x)^{-q\beta}$$

which would imply

$$\int_{\Omega} G_{\Omega}(x, y) f(y, u(y)) dy = +\infty, \quad x \in \Omega,$$

which means that the problem is not solvable.

3.6 COMPLETE BLOW-UP: PROOF OF THEOREM 1.18

Consider a nondecreasing sequence $\{f_k\}_{k \in \mathbb{N}}$ of bounded functions such that $f_k \uparrow f$ pointwisely as $k \uparrow +\infty$, as in Definition 1.17. Let us first prove the theorem in the case of null boundary data. The first claim is that

$$\int_{\Omega} f_k(x, u_k(x)) \delta(x)^s dx \xrightarrow{k \uparrow +\infty} +\infty.$$

Suppose by contradiction that the sequence of integrals is bounded by a constant C . Consider an increasing sequence of nonnegative $\{\psi_n\}_{n \in \mathbb{N}} \subseteq C_c^{\infty}(\Omega)$ such that $\psi_n \uparrow 1$ in Ω and pick $\phi_n \in \mathcal{T}(\Omega)$ in such a way that $(-\Delta)^s \phi_n = \psi_n$ holds in Ω . Then

$$\int_{\Omega} u_k \psi_n = \int_{\Omega} f_k(x, u_k) \phi_n \leq c \int_{\Omega} f_k(x, u_k(x)) \delta(x)^s dx$$

for some constant $c > 0$ not depending on n , see [73, Proposition 1.2]. By letting $n \uparrow +\infty$, we deduce that u_k is a bounded sequence in $L^1(\Omega)$. Take now \underline{u}_k as the minimal solution to the k -th nonlinear problem: since $(-\Delta)^s \underline{u}_{k+1} = f_{k+1}(x, \underline{u}_{k+1}) \geq f_k(x, \underline{u}_{k+1})$ then \underline{u}_k is an increasing sequence and it admits a pointwise limit u . This u is limit also in the L^1 -norm, since \underline{u}_k is bounded in this norm. But then, for any $\phi \in \mathcal{T}(\Omega)$, we have by the Monotone Convergence Theorem

$$\int_{\Omega} u (-\Delta)^s \phi = \lim_{k \uparrow +\infty} \int_{\Omega} \underline{u}_k (-\Delta)^s \phi = \lim_{k \uparrow +\infty} \int_{\Omega} f_k(x, \underline{u}_k) \phi = \int_{\Omega} f(x, u) \phi,$$

that is $u \in L^1(\Omega)$ would be a weak solution, a contradiction.

Our second claim is that

$$\underline{u}_k(x) \geq c \left[\int_{\Omega} f_k(y, \underline{u}_k(y)) \delta(y)^s dy \right] \delta(x)^s,$$

for some constant $c > 0$ independent of k . To do this, we exploit (1.14). Call $\Omega_1(x) = \{y \in \Omega : |y - x| \leq \delta(x)\delta(y)\}$, $\Omega_2(x) = \{y \in \Omega : |y - x| > \delta(x)\delta(y)\}$ and $d(\Omega) = \sup\{|y - x| : x, y \in \Omega\}$:

$$\begin{aligned} \underline{u}_k(x) &= \int_{\Omega} G_{\Omega}(x, y) f_k(y, \underline{u}_k(y)) dy \\ &\geq c_2 \int_{\Omega} \left[|x - y|^{2s} \wedge \delta(x)^s \delta(y)^s \right] f_k(y, \underline{u}_k(y)) \frac{dy}{|x - y|^N} \\ &= c_2 \int_{\Omega_1(x)} f_k(y, \underline{u}_k(y)) \frac{dy}{|x - y|^{N-2s}} + c_2 \delta(x)^s \int_{\Omega_2(x)} \delta(y)^s f_k(y, \underline{u}_k(y)) \frac{dy}{|x - y|^N} \\ &\geq \frac{c_2}{d(\Omega)^{2s}} \int_{\Omega_1(x)} \delta(x)^s \delta(y)^s f_k(y, \underline{u}_k(y)) \frac{dy}{|x - y|^{N-2s}} + \frac{c_2}{d(\Omega)^N} \delta(x)^s \int_{\Omega_2(x)} \delta(y)^s f_k(y, \underline{u}_k(y)) dy \\ &\geq \frac{c_2}{d(\Omega)^N} \delta(x)^s \int_{\Omega} \delta(y)^s f_k(y, \underline{u}_k(y)) dy. \end{aligned}$$

Let now $k \uparrow +\infty$: by the monotone pointwise convergence of f_k to f and by the monotonicity of $\{\underline{u}_k\}_{k \in \mathbb{N}}$ (as noticed above), the last term in the inequalities above converges to $+\infty$, and we see then how we have complete blow-up.

In the case of nonhomogeneous boundary conditions, we consider the s -harmonic function u_0 induced by data g and h , and we denote by $F(x, t) = f(x, u_0(x) + t)$ for $x \in \Omega$, $t \geq 0$. By hypothesis we have then that there is no weak solution to

$$\begin{cases} (-\Delta)^s v = F(x, v) & \text{in } \Omega, \\ v = 0 & \text{in } \mathcal{C}\Omega, \\ Ev = 0 & \text{on } \partial\Omega. \end{cases}$$

Since any monotone approximation on f is also a monotone approximation of F , then there is complete blow-up in the problem for v and this bears the complete blow-up for the problem on u .

A Keller-Osserman condition for $(-\Delta)^s$

In this Chapter we are going to provide the proofs of the results listed in Section 1.4. Recall the list of assumptions we have set ourselves in.

4.1 PRELIMINARIES

Hypothesis (1.20) implies that $f(t)t^{-1-M}$ is monotone decreasing and $f(t)t^{-1-m}$ is monotone increasing, since

$$\frac{d}{dt} \frac{f(t)}{t^{1+M}} = \frac{1}{t^{1+M}} \left(f'(t) - (1+M) \frac{f(t)}{t} \right) \leq 0, \quad \frac{d}{dt} \frac{f(t)}{t^{1+m}} = \frac{1}{t^{1+m}} \left(f'(t) - (1+m) \frac{f(t)}{t} \right) \geq 0 :$$

we write this monotonicity conditions as

$$c^{1+m} f(t) \leq f(ct) \leq c^{1+M} f(t), \quad c > 1, t > 0. \quad (4.1)$$

The function F satisfies two inequalities similar to (1.20):

$$2+m \leq \frac{t f(t)}{F(t)} \leq 2+M, \quad (4.2)$$

indeed by integrating (1.20) we deduce

$$(1+m)F(t) \leq \int_0^t \tau f'(\tau) d\tau = t f(t) - F(t).$$

Let $\psi = \phi^{-1}$ be the inverse of ϕ , so that

$$v = \int_{\psi(v)}^{+\infty} \frac{dt}{\sqrt{F(t)}}, \quad v \geq 0. \quad (4.3)$$

The function ψ is decreasing and $\psi(v) \uparrow +\infty$ as $v \downarrow 0$. Moreover, by Remark 1.19 and (4.2), for $u > 0$ and some $y \in (u, +\infty)$

$$\frac{\phi(u)}{u|\phi'(u)|} = \frac{\sqrt{F(u)}}{u} \int_u^{+\infty} \frac{dt}{\sqrt{F(t)}} = \frac{-\frac{1}{\sqrt{F(y)}}}{\frac{1}{\sqrt{F(y)}} - \frac{y f(y)}{2F(y)^{3/2}}} = \frac{1}{\frac{y f(y)}{2F(y)} - 1} \quad \left\{ \begin{array}{l} \geq \frac{2}{M} \\ \leq \frac{2}{m} \end{array} \right.$$

which in turn says that it holds, by setting $v = \phi(u)$,

$$\frac{2}{M} \leq \frac{v|\psi'(v)|}{\psi(v)} \leq \frac{2}{m}, \quad (4.4)$$

and one can also prove

$$\psi(cv) \leq c^{-2/M} \psi(v), \quad c \in (0, 1), \quad v > 0. \quad (4.5)$$

as we have done for (4.1) above. Also, by (4.2) and (4.4),

$$\frac{v^2 \psi''(v)}{\psi(v)} = \frac{v^2 f(\psi(v))}{2 \psi(v)} \asymp \frac{v^2 F(\psi(v))}{\psi(v)^2} = \frac{v^2 \psi'(v)^2}{\psi(v)^2} \asymp 1. \quad (4.6)$$

4.2 CONSTRUCTION OF A SUPERSOLUTION

In this paragraph we prove the key point for the proof of Theorems 1.21 and 1.23, that is we build a supersolution to both problems by handling the function U defined in (4.7) below.

Since by assumption $\partial\Omega \in C^2$, the function $\text{dist}(x, \partial\Omega)$ is C^2 in an open strip around the boundary, except on $\partial\Omega$ itself. Consider a positive function $\delta(x)$ which is obtained by extending $\text{dist}(x, \partial\Omega)$ smoothly to $\mathbb{R}^N \setminus \partial\Omega$. Define

$$U(x) = \psi(\delta(x)^s), \quad x \in \mathbb{R}^N. \quad (4.7)$$

LEMMA 4.1 | *The function U defined in (4.7) is in $L^1(\Omega)$.*

Proof. Since both ψ and δ^s are continuous in Ω , then $U \in L^1_{loc}(\Omega)$. Fix $\delta_0 > 0$ small and consider $\Omega_0 = \{x \in \Omega : \delta(x) < \delta_0\}$. We have (using once the coarea formula)

$$\int_{\Omega_0} \psi(\delta(x)^s) dx \leq C \int_0^{\delta_0} \psi(t^s) dt :$$

apply now the transformation $\psi(t^s) = \eta$ to get

$$\int_{\Omega_0} U(x) dx \leq C \int_{\eta_0}^{+\infty} \eta \phi(\eta)^{(1-s)/s} |\phi'(\eta)| d\eta$$

where, by Remark 1.19,

$$|\phi'(\eta)| \asymp \frac{1}{\sqrt{\eta f(\eta)}} \quad \text{and} \quad \phi(\eta) \asymp \sqrt{\frac{\eta}{f(\eta)}},$$

therefore

$$\int_{\Omega_0} U(x) dx \leq C \int_{\eta_0}^{+\infty} \left(\frac{\eta}{f(\eta)} \right)^{\frac{1}{2s}} d\eta$$

which is finite by (1.23). □

PROPOSITION 4.2 | *The function U defined in (4.7) satisfies for some C , $\delta_0 > 0$*

$$(-\Delta)^s U \geq -Cf(U), \quad \text{in } \Omega_{\delta_0} = \{x \in \Omega : \delta(x) < \delta_0\}. \quad (4.8)$$

Proof. We start by writing, for $x \in \Omega$

$$\frac{(-\Delta)^s U(x)}{C_{N,s}} = PV \int_{\Omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy + \int_{\mathbb{R}^N \setminus \Omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy. \quad (4.9)$$

Let us begin with an estimate for

$$PV \int_{\Omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy.$$

Split the integral into

$$\int_{\Omega_1} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy + PV \int_{\Omega_2} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy + \int_{\Omega_3} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy$$

where we have set

$$\begin{aligned}\Omega &= \Omega_1 \cup \Omega_2 \cup \Omega_3, \text{ with: } \Omega_1 = \left\{ y \in \Omega : \delta(y) > \frac{3}{2}\delta(x) \right\} \\ \Omega_2 &= \left\{ y \in \Omega : \frac{1}{2}\delta(x) \leq \delta(y) \leq \frac{3}{2}\delta(x) \right\} \\ \Omega_3 &= \left\{ y \in \Omega : \delta(y) < \frac{1}{2}\delta(x) \right\}.\end{aligned}$$

In Ω_1 we have in particular $\delta(y) > \delta(x)$ so that, since ψ decreasing function, the first integral contributes by a positive quantity.

Now, let us turn to integrals on Ω_2 and Ω_3 . Set $x = \theta + \delta(x)\nabla\delta(x)$, $\theta \in \partial\Omega$: up to a rotation and a translation, we can suppose that $\theta = 0$ and $\nabla\delta(x) = e_N$. By compactness, $\partial\Omega$ can be covered by a finite number of open portions $\Gamma_j \subset \partial\Omega$, $j = 1, \dots, n$. For any $j = 1, \dots, n$, the function $\eta \mapsto \text{dist}(\eta, \partial\Omega \setminus \Gamma_j)$ is continuous in $\partial\Omega$ and so is $\eta \mapsto \max_j \text{dist}(\eta, \partial\Omega \setminus \Gamma_j)$: there is a point $\eta^* \in \partial\Omega$ where $\eta \mapsto \max_j \text{dist}(\eta, \partial\Omega \setminus \Gamma_j)$ attains its minimum. Such a minimum cannot be 0 because η^* belongs at least to one of the Γ_j . This implies that for any $\eta \in \partial\Omega$ there exists $i \in \{1, \dots, n\}$ for which

$$\text{dist}(\eta, \partial\Omega \setminus \Gamma_i) \geq \max_j \text{dist}(\eta^*, \partial\Omega \setminus \Gamma_j) \quad (4.10)$$

and this in particular implies $\eta \in \Gamma_i$. Let Γ be a neighbourhood of 0 on $\partial\Omega$ chosen from $\{\Gamma_j\}_{j=1}^n$ and for which (4.10) is fulfilled. Let also

$$\omega = \{y \in \mathbb{R}^N : y = \eta + \delta(y)\nabla\delta(y), \eta \in \Gamma\}.$$

The set $\Gamma \subset \partial\Omega$ can be described via as the graph of a C^2 function

$$\begin{aligned}\gamma : B'_r(0) &\subseteq \mathbb{R}^{N-1} \longrightarrow \mathbb{R} \\ \eta' &\longmapsto \gamma(\eta') \quad \text{s.t. } \eta = (\eta', \gamma(\eta')) \in \Gamma\end{aligned}$$

satisfying $\gamma(0) = |\nabla\gamma(0)| = 0$.

The integration on $(\Omega_2 \cup \Omega_3) \setminus \omega$ is lower order with respect to the one on $(\Omega_2 \cup \Omega_3) \cap \omega$ since in the latter we have the singular in x to deal with, while in the former $|x - y|$ is a quantity bounded below independently on x . Indeed when $y \in (\Omega_2 \cup \Omega_3) \setminus \omega$

$$|x - y| \geq |\eta + \delta(y)\nabla\delta(y) - \delta(x)| \geq |\eta| - \delta(y) - \delta(x) \geq \text{dist}(0, \partial\Omega \setminus \Gamma) - \frac{5}{2}\delta(x)$$

where $\delta(x)$ is small and the first addend is bounded uniformly in x by (4.10).

We are left with:

$$C \cdot PV \int_{\Omega_2 \cap \omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy + C \int_{\Omega_3 \cap \omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy.$$

Let us split the remainder of the estimate in steps.

First step: the distance between x and y . We claim that there exists $c > 0$ such that

$$\begin{aligned}|x - y|^2 &\geq c(|\delta(x) - \delta(y)|^2 + |\eta'|^2), \quad y \in (\Omega_2 \cup \Omega_3) \cap \omega, \\ y &= \eta + \delta(y)\nabla\delta(y), \quad \eta = (\eta', \gamma(\eta')).\end{aligned} \quad (4.11)$$

Since in our set of coordinates $x = \delta(x)e_N$, we can write

$$\begin{aligned}|x - y|^2 &= |\delta(x)e_N - \delta(y)e_N + \delta(y)e_N - y_N e_N - y'|^2 \geq \\ &\geq |\delta(x) - \delta(y)|^2 - 2|\delta(x) - \delta(y)| \cdot |\delta(y) - y_N| + |\delta(y) - y_N|^2 + |y'|^2.\end{aligned}$$

We concentrate our attention on $|\delta(y) - y_N|$: the idea is to show that this is a small quantity; indeed, in the particular case when Γ lies on the hyperplane $y_N = 0$, this quantity is actually zero. As in the

definition of ω , we let $y = \eta + \delta(y)\nabla\delta(y)$ and $\eta = (\eta', \gamma(\eta')) \in \Gamma$: thus $y_N = \gamma(\eta') + \delta(y)\langle \nabla\delta(y), e_N \rangle$ where $\nabla\delta(y)$ is the inward unit normal to $\partial\Omega$ at the point η , so that

$$\nabla\delta(y) = \frac{(-\nabla\gamma(\eta'), 1)}{\sqrt{|\nabla\gamma(\eta')|^2 + 1}}$$

and

$$y' = \eta' - \frac{\delta(y)\nabla\gamma(\eta')}{\sqrt{|\nabla\gamma(\eta')|^2 + 1}}, \quad y_N = \gamma(\eta') + \frac{\delta(y)}{\sqrt{|\nabla\gamma(\eta')|^2 + 1}}. \quad (4.12)$$

Now, since $y \in \Omega_2 \cup \Omega_3$, it holds $|\delta(x) - \delta(y)| \leq \delta(x)$ and

$$|\delta(y) - y_N| \leq |\gamma(\eta')| + \delta(y) \left(1 - \frac{1}{\sqrt{|\nabla\gamma(\eta')|^2 + 1}} \right) \leq C|\eta'|^2 + 2C\delta(x)|\eta'|^2$$

where, in this case, $C = \|\gamma\|_{C^2(B_r)}$ depends only on the geometry of $\partial\Omega$ and not on x . By (4.12), we have

$$|\eta'|^2 \leq 2|y'|^2 + \frac{2\delta(y)^2 |\nabla\gamma(\eta')|^2}{|\nabla\gamma(\eta')|^2 + 1} \leq 2|y'|^2 + 2C\delta(y)^2 |\eta'|^2 \leq 2|y'|^2 + C\delta(x)^2 |\eta'|^2,$$

so that $|\eta'|^2 \leq C|y'|^2$ when $\delta(x)$ is small enough. Finally

$$\begin{aligned} |x - y|^2 &\geq |\delta(x) - \delta(y)|^2 + |y'|^2 - 2|\delta(x) - \delta(y)| \cdot |\delta(y) - y_N| \geq \\ &\geq |\delta(x) - \delta(y)|^2 + c|\eta'|^2 - 2C\delta(x)|\eta'|^2, \end{aligned}$$

where, again, $C = \|\gamma\|_{C^2(B_r)}$ and (4.11) is proved provided x is close enough to $\partial\Omega$.

Second step: integration on $\Omega_2 \cap \omega$. Using the regularity of ψ and δ we write

$$\psi(\delta(x)^s) - \psi(\delta(y)^s) \geq \nabla(\psi \circ \delta^s)(x) \cdot (x - y) - \|D^2(\psi \circ \delta^s)\|_{L^\infty(\Omega_2 \cap \omega)} |x - y|^2$$

where

$$D^2(\psi \circ \delta^s) = \frac{s\psi'(\delta^s)}{\delta^{1-s}} D^2\delta + \frac{s^2\psi''(\delta^s)}{\delta^{2-2s}} \nabla\delta \otimes \nabla\delta + \frac{s(s-1)\psi'(\delta^s)}{\delta^{2-s}} \nabla\delta \otimes \nabla\delta$$

so that

$$\|D^2(\psi \circ \delta^s)\|_{L^\infty(\Omega_2 \cap \omega)} \leq C \left\| \frac{\psi'(\delta^s)}{\delta^{1-s}} \right\|_{L^\infty(\Omega_2 \cap \omega)} + C \left\| \frac{\psi''(\delta^s)}{\delta^{2-2s}} \right\|_{L^\infty(\Omega_2 \cap \omega)} + C \left\| \frac{\psi'(\delta^s)}{\delta^{2-s}} \right\|_{L^\infty(\Omega_2 \cap \omega)}.$$

By definition of Ω_2 and by (4.5) we can control the sup-norm by the value at x :

$$\begin{aligned} \|D^2(\psi \circ \delta^s)\|_{L^\infty(\Omega_2 \cap \omega)} &\leq C \frac{|\psi'(\delta(x)^s)|}{\delta(x)^{1-s}} + C \frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}} + C \frac{|\psi'(\delta(x)^s)|}{\delta(x)^{2-s}} \leq \\ &\leq C \frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}} + C \frac{|\psi'(\delta(x)^s)|}{\delta(x)^{2-s}} \end{aligned}$$

and using equation (4.6) we finally get

$$\|D^2(\psi \circ \delta^s)\|_{L^\infty(\Omega_2 \cap \omega)} \leq C \frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}}.$$

If we now retrieve the whole integral and exploit (4.11)

$$\begin{aligned} PV \int_{\Omega_2 \cap \omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy &\geq -C \frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}} \int_{\Omega_2 \cap \omega} \frac{dy}{|x - y|^{N+2s-2}} \geq \\ &\geq -C \frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}} \int_{\Omega_2 \cap \omega} \frac{dy}{(|\delta(x) - \delta(y)|^2 + |\eta|^2)^{(N+2s-2)/2}}. \end{aligned}$$

We focus our attention on the integral on the right-hand side: by the coarea formula

$$\begin{aligned}
& \int_{\Omega_2 \cap \omega} \frac{dy}{(|\delta(x) - \delta(y)|^2 + |\eta|^2)^{(N+2s-2)/2}} = \\
&= \int_{\delta(x)/2}^{3\delta(x)/2} dt \int_{\{\delta(y)=t\} \cap \omega} \frac{d\mathcal{H}^{N-1}(\eta)}{(|\delta(x) - t|^2 + |\eta|^2)^{(N+2s-2)/2}} \\
&\leq C \int_{\delta(x)/2}^{3\delta(x)/2} dt \int_{B_r} \frac{d\eta'}{(|\delta(x) - t|^2 + |\eta'|^2)^{(N+2s-2)/2}} \\
&\leq C \int_{\delta(x)/2}^{3\delta(x)/2} dt \int_0^r \frac{\rho^{N-2}}{(|\delta(x) - t|^2 + \rho^2)^{(N+2s-2)/2}} d\rho \\
&\leq C \int_{\delta(x)/2}^{3\delta(x)/2} dt \int_0^r \frac{\rho}{(|\delta(x) - t|^2 + \rho^2)^{(2s+1)/2}} d\rho \leq C \int_{\delta(x)/2}^{3\delta(x)/2} \frac{dt}{|t - \delta(x)|^{2s-1}}.
\end{aligned}$$

We can retrieve now the chain of inequalities we stopped above:

$$\int_{\Omega_3 \cap \omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy \geq -C \frac{\psi''(\delta(x)^s)}{\delta(x)^{2-2s}} \int_{\delta(x)/2}^{3\delta(x)/2} \frac{dt}{|t - \delta(x)|^{-1+2s}} \geq -C \psi''(\delta(x)^s).$$

Third step: integration on $\Omega_3 \cap \omega$. We use (4.11) once again:

$$\begin{aligned}
& \int_{\Omega_3 \cap \omega} \frac{\psi(\delta(x)^s) - \psi(\delta(y)^s)}{|x - y|^{N+2s}} dy \geq \\
&\geq - \int_{\Omega_3 \cap \omega} \frac{\psi(\delta(y)^s)}{|x - y|^{N+2s}} dy \geq -C \int_{\Omega_3 \cap \omega} \frac{\psi(\delta(y)^s)}{(|\delta(x) - \delta(y)|^2 + |\eta'|^2)^{\frac{N+2s}{2}}} dy \\
&\geq -C \int_0^{\delta(x)/2} \frac{\psi(t^s)}{(\delta(x) - t)^{1+2s}} dt \geq -\frac{C}{\delta(x)^{1+2s}} \int_0^{\delta(x)/2} \psi(t^s) dt.
\end{aligned}$$

The term we have obtained is of the same order of $\delta(x)^{-2s}\psi(\delta(x)^s)$, by (4.4):

$$\int_0^{\delta(x)/2} \psi(t^s) dt \asymp \int_0^{\delta(x)/2} t^s \psi'(t^s) dt = \frac{\delta(x)}{2s} \psi\left(\frac{\delta(x)^s}{2^s}\right) - \frac{1}{s} \int_0^{\delta(x)/2} \psi(t^s) dt$$

so that

$$\int_0^{\delta(x)/2} \psi(t^s) dt \asymp \delta(x) \psi(\delta(x)^s) = \delta(x)^{1+2s} \cdot \frac{\psi(\delta(x)^s)}{\delta(x)^{2s}}. \quad (4.13)$$

Recall now that $\psi(\delta(x)^s)\delta(x)^{-2s}$ is in turn of the same size of $\psi''(\delta(x)^s)$ by (4.6).

Fourth step: the outside integral in (4.9). We focus now our attention on

$$\int_{C\Omega} \frac{\psi(\delta(y)^s) - \psi(\delta(x)^s)}{|x - y|^{N+2s}} dy.$$

First, by using the monotonicity of ψ , we write

$$\begin{aligned}
\int_{C\Omega} \frac{\psi(\delta(y)^s) - \psi(\delta(x)^s)}{|x - y|^{N+2s}} dy &\leq \int_{\{y \in C\Omega: \delta(y) < \delta(x)\} \cap \omega} \frac{\psi(\delta(y)^s) - \psi(\delta(x)^s)}{|x - y|^{N+2s}} dy + \\
&\quad + \int_{\{y \in C\Omega: \delta(y) < \delta(x)\} \setminus \omega} \frac{\psi(\delta(y)^s) - \psi(\delta(x)^s)}{|x - y|^{N+2s}} dy.
\end{aligned}$$

The second integral gives

$$\int_{\{y \in C\Omega: \delta(y) < \delta(x)\} \setminus \omega} \frac{\psi(\delta(y)^s) - \psi(\delta(x)^s)}{|x - y|^{N+2s}} dy \leq C \|\psi(\delta^s)\|_{L^1(\mathbb{R}^N)}$$

because the distance between x and y is bounded there. Again we point out that

$$\begin{aligned}
& \int_{\{\delta(y) < \delta(x)\} \cap \omega} \frac{\psi(\delta(y)^s) - \psi(\delta(x)^s)}{|x - y|^{N+2s}} dy \leq C \int_0^{\delta(x)} \frac{\psi(t^s) - \psi(\delta(x)^s)}{|\delta(x) + t|^{1+2s}} dt \leq \\
& \leq C \int_0^{\delta(x)/2} \frac{\psi(t^s)}{|\delta(x) + t|^{1+2s}} dt + C \int_{\delta(x)/2}^{\delta(x)} \frac{\psi(t^s)}{|\delta(x) + t|^{1+2s}} dt \\
& \leq C \delta(x)^{-1-2s} \int_0^{\delta(x)/2} \psi(t^s) dt + C \psi\left(\frac{\delta(x)^s}{2^s}\right) \int_{\delta(x)/2}^{\delta(x)} (\delta(x) + t)^{-1-2s} dt \\
& \leq C \delta(x)^{-1-2s} \int_0^{\delta(x)} \psi(t^s) dt + C \psi(\delta(x)^s) \delta(x)^{-2s}
\end{aligned}$$

which is of the order of $\psi''(\delta(x)^s)$, by (4.13) and (4.6).

Conclusion. We have proved that for $\delta(x)$ sufficiently small

$$(-\Delta)^s U(x) \geq -C\psi''(\delta(x)^s).$$

Recall now that $\psi''(\delta^s) = f(\psi \circ \delta^s)/2$ and $U = \psi \circ \delta^s$ in Ω , so that

$$(-\Delta)^s U \geq -Cf(U)$$

holds in a neighbourhood of $\partial\Omega$. □

4.3 EXISTENCE

LEMMA 4.3 | *If the nonlinear term f satisfies the growth condition (1.25) then the function U defined in (4.7) satisfies*

$$\lim_{x \rightarrow \partial\Omega} \delta(x)^{1-s} U(x) = +\infty.$$

Proof. Write

$$\liminf_{x \rightarrow \partial\Omega} \delta(x)^{1-s} \psi(\delta(x)^s) = \liminf_{u \uparrow +\infty} u \phi(u)^{\frac{1-s}{s}}.$$

Such a limit is $+\infty$ if and only if

$$\liminf_{u \uparrow +\infty} u^{\frac{s}{1-s}} \int_u^{+\infty} \frac{dt}{\sqrt{2F(t)}} = +\infty.$$

If we use the L'Hôpital's rule to

$$\frac{\int_u^{+\infty} \frac{dt}{\sqrt{2F(t)}}}{u^{-s/(1-s)}}$$

we get the ratio $u^{\frac{1}{1-s}}/\sqrt{2F(u)}$ and applying once again the L'Hôpital's rule, this time to $u^{\frac{2}{1-s}}/F(u)$, we get $u^{\frac{1+s}{1-s}}/f(u)$ which diverges by hypothesis (1.25). Indeed, since f is increasing,

$$u^{-\frac{1+s}{1-s}} f(u) = f(u) \cdot \frac{1-s}{1+s} \int_u^{+\infty} t^{-2/(1-s)} dt \leq \int_u^{+\infty} f(t) t^{-2/(1-s)} dt \xrightarrow{u \uparrow +\infty} 0.$$

□

LEMMA 4.4 | *Let $v : \mathbb{R}^N \rightarrow \mathbb{R}$ be a function which satisfies $(-\Delta)^s v \in C(\Omega)$. If there exist $C, \delta_0 > 0$ such that*

$$(-\Delta)^s v \geq -Cf(v) \quad \text{in } \Omega_{\delta_0} := \{x \in \Omega : \delta(x) < \delta_0\}$$

then there exists $\bar{u} \geq v$ such that $(-\Delta)^s \bar{u} \geq -f(\bar{u})$ throughout Ω .

Proof. Define $\xi : \mathbb{R}^N \rightarrow \mathbb{R}$ as the solution to

$$\begin{cases} (-\Delta)^s \xi = 1 & \text{in } \Omega \\ \xi = 0 & \text{in } \mathcal{C}\Omega \\ E\xi = 0 & \text{on } \partial\Omega \end{cases} \quad (4.14)$$

and consider $\bar{u} = \mu v + \lambda \xi$, where $\mu, \lambda \geq 1$. If $C \in (0, 1]$ then $(-\Delta)^s v \geq -f(v)$ in Ω_{δ_0} , so choose $\mu = 1$. If $C > 1$, then choose $\mu = C^{1/M} > 1$ in order to have in Ω_{δ_0}

$$\begin{aligned} (-\Delta)^s \bar{u} + f(\bar{u}) &= (-\Delta)^s (\mu v + \lambda \xi) + f(\mu v + \lambda \xi) \geq -\mu C f(v) + f(\mu v) \geq \\ &\geq (-\mu C + \mu^{1+M}) f(v) = 0 \end{aligned}$$

where we have heavily used the positivity of ξ and (4.1). Now, since $(-\Delta)^s v \in C(\overline{\Omega \setminus \Omega_{\delta_0}})$ we can choose $\lambda = \mu \|(-\Delta)^s v\|_{L^\infty(\Omega \setminus \Omega_{\delta_0})}$ so that also in $\Omega \setminus \Omega_{\delta_0}$

$$(-\Delta)^s \bar{u} = (-\Delta)^s (\mu v + \lambda \xi) = \mu (-\Delta)^s v + \lambda \geq 0 \geq -f(\bar{u}).$$

□

Collecting the information so far, we have that Lemmas 4.1, 4.4 and 4.3 fully prove the following theorems.

THEOREM 4.5 | *If the nonlinear term f satisfies (1.18), (1.20), (1.22) and the growth condition (1.25), then there exists a function \bar{u} supersolution to (1.26). Moreover*

$$\bar{u} = \mu \psi(\delta^s) + \lambda \xi, \quad \text{in } \Omega$$

where ξ is the solution of (4.14), $\lambda > 0$, $\mu = \max\{1, C^{1/M}\}$ where $C > 0$ is the constant in (4.8) and $M > 0$ the one in (1.20).

THEOREM 4.6 | *If the nonlinear term f satisfies (1.18), (1.20), (1.22), then there exists a function \bar{u} supersolution to (1.29). Moreover*

$$\bar{u} = \mu \psi(\delta^s) + \lambda \xi, \quad \text{in } \Omega$$

where ξ is the solution of (4.14), $\lambda > 0$, $\mu = \max\{1, C^{1/M}\}$ where $C > 0$ is the constant in (4.8) and $M > 0$ the one in (1.20).

4.3.1 Proof of Theorem 1.21

Theorems 4.5 bears as a consequence the following. Build the sequence of weak solutions to problems

$$\begin{cases} (-\Delta)^s u_k = -f(u_k) & \text{in } \Omega \\ u_k = 0 & \text{in } \mathcal{C}\Omega \\ E u_k = k & \text{on } \partial\Omega, \quad k \in \mathbb{N}. \end{cases} \quad (4.15)$$

The existence of any u_k can be proved as in Theorem 1.13, in view of hypothesis (1.25), since it implies

$$\int_0^{\delta_0} f(\delta^{s-1}) \delta^s d\delta < +\infty.$$

We need here auxiliary regularity results that will be proved in the next chapter. Using Lemma 5.2 and Corollary 5.3, we deduce that $u_k \in C^{2s+\varepsilon}(\Omega)$ and

$$(-\Delta)^s u_k(x) = -f(u_k(x)), \quad \text{for any } x \in \Omega$$

in a pointwise sense.

Step 0: $\{u_k\}_{k \in \mathbb{N}}$ is increasing with k . Consider

$$\begin{cases} (-\Delta)^s(u_{k+1} - u_k) = f(u_k) - f(u_{k+1}) & \text{in } \Omega \\ u_{k+1} - u_k = 0 & \text{in } \mathcal{C}\Omega \\ E(u_{k+1} - u_k) = 1 & \text{on } \partial\Omega. \end{cases}$$

Call $\Omega_k^+ := \{u_k > u_{k+1}\}$ which satisfies $\overline{\Omega_k^+} \subset \Omega$ as it is implied by the singular boundary trace. In $\overline{\Omega_k^+} \subset \Omega$ the difference $u_k - u_{k+1}$ attains its maximum at some point x_k : at this point we must have $(-\Delta)^s(u_k - u_{k+1})(x_k) > 0$ because it is also a global maximum. But at the same time $f(u_{k+1}(x_k)) - f(u_k(x_k)) \leq 0$, because f is increasing, proving that Ω_k^+ must be empty.

Step 1: $\{u_k\}_{k \in \mathbb{N}}$ has a pointwise limit. Any u_k lies below \bar{u} : since $E(\bar{u} - u_k) > 0$, there exists a compact set $U_k \subset \Omega$ for which $u_k \leq \bar{u}$ in $\Omega \setminus U_k$. Inside U_k we have

$$\begin{cases} (-\Delta)^s(\bar{u} - u_k) \geq f(u_k) - f(\bar{u}) & \text{in } U_k \\ \bar{u} - u_k \geq 0 & \text{in } \mathcal{C}U_k \\ E_{\partial U_k}(\bar{u} - u_k) = 0 & \text{on } \partial U_k \end{cases}$$

in a pointwise sense, where $u_k, \bar{u} \in C^{2s+\varepsilon}(\overline{U_k})$. Then an argument similar to the one in *Step 0* yields $u_k \leq \bar{u}$ also in U_k .

Finally, $\{u_k\}_{k \in \mathbb{N}}$ is increasing and pointwisely bounded by \bar{u} throughout Ω . This entails that

$$u(x) := \lim_{k \uparrow +\infty} u_k(x)$$

is well-defined and finite for any $x \in \Omega$. Also, $0 \leq u \leq \bar{u}$ in Ω and since $\bar{u} \in L^1(\Omega)$ by Lemma 4.1, then $u \in L^1(\Omega)$.

Step 2: $u \in C(\Omega)$. Fix any compact $D \subset \Omega$ and choose a $c > 0$ for which $\delta(x) > 2c$ for any $x \in D$. Let $\tilde{D} := \{y \in \Omega : \delta(y) > c\}$. For any $k, j \in \mathbb{N}$ it holds

$$(-\Delta)^s(u_{k+j} - u_k) = f(u_k) - f(u_{k+j}) \leq 0, \quad \text{in } \tilde{D}$$

and therefore

$$0 \leq u_{k+j}(x) - u_k(x) \leq \int_{\mathcal{C}\tilde{D}} P_{\tilde{D}}(x, y) [u_{k+j}(y) - u_k(y)] dy$$

where $P_{\tilde{D}}(x, y)$ is the Poisson kernel associated to \tilde{D} , which satisfies (see [27, Theorem 2.10])

$$P_{\tilde{D}}(x, y) \leq \frac{C \delta_{\tilde{D}}(x)^s}{\delta_{\tilde{D}}(y)^s |x - y|^N}, \quad x \in \tilde{D}, y \in \mathcal{C}\tilde{D}.$$

When $x \in D \subset \tilde{D}$ one has $|x - y| > c$ for any $y \in \mathcal{C}\tilde{D}$, and therefore

$$0 \leq u_{k+j}(x) - u_k(x) \leq C \int_{\mathcal{C}\tilde{D}} \frac{u_{k+j}(y) - u_k(y)}{\delta_{\tilde{D}}(y)^s} dy \leq C \int_{\mathcal{C}\tilde{D}} \frac{u(y) - u_k(y)}{\delta_{\tilde{D}}(y)^s} dy$$

where the last integral converges by Monotone Convergence to 0 independently on x . This means the convergence $u_k \rightarrow u$ is uniform on compact subsets and since $\{u_k\}_{k \in \mathbb{N}} \subset C(\Omega)$ (cf. the proof of Theorem 1.13, Section 3.3), then also $u \in C(\Omega)$.

Step 3: $u \in C^2(\Omega)$. This is a standard bootstrap argument using the elliptic regularity in [78, Propositions 2.8 and 2.9].

Step 4: u solves (1.26) in a pointwise sense. The function $(-\Delta)^s u(x)$ is well-defined for any $x \in \Omega$ because $u \in C^2(\Omega) \cap L^1(\mathbb{R}^N)$. Using the regularity results in [78, Propositions 2.8 and 2.9], we have

$$(-\Delta)^s u = \lim_{k \uparrow +\infty} (-\Delta)^s u_k = - \lim_{k \uparrow +\infty} f(u_k) = -f(u).$$

Also, $\delta^{1-s} u \geq \delta^{1-s} u_k$ holds in Ω for any $k \in \mathbb{N}$. Therefore, for any $k \in \mathbb{N}$,

$$\liminf_{x \rightarrow \partial\Omega} \delta(x)^{1-s} u(x) \geq \lim_{x \rightarrow \partial\Omega} \delta(x)^{1-s} u_k(x) \geq \lambda E u_k = \lambda k$$

for some constant $\lambda > 0$ depending on Ω and not on k . This entails

$$\lim_{x \rightarrow \partial\Omega} \delta(x)^{1-s} u(x) = +\infty$$

and completes the proof of Theorem 1.21.

REMARK 4.7 | The proof of Theorem 1.23 is alike. Indeed, in the same way, the sequence of solutions to

$$\begin{cases} (-\Delta)^s u_k = -f(u_k) & \text{in } \Omega \\ u_k = g_k := \min\{k, g\} & \text{in } \mathcal{C}\Omega, \ k \in \mathbb{N} \\ E u_k = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.16)$$

approaches a solution of problem (1.29) which lies below the supersolution provided by Theorem 4.6.

4.4 EXAMPLES

EXAMPLE 4.8: Power nonlinearity | Let us consider $f(t) = t^p$, for $p > 1$. In this case

$$\frac{t f'(t)}{f(t)} = p.$$

The function ϕ reads as (cf. (1.21))

$$\phi(u) = \int_u^{+\infty} \sqrt{\frac{p+1}{2}} t^{-\frac{p+1}{2}} dt = \sqrt{\frac{2(p+1)}{p-1}} u^{\frac{1-p}{2}}$$

and hypothesis (1.22) can then be written

$$\int_u^{+\infty} \eta^{\frac{1-p}{2s}} d\eta < +\infty$$

that holds if and only if $p > 1 + 2s$. On the other hand hypothesis (1.25) becomes

$$p - \frac{2}{1-s} < -1, \quad \text{i.e. } p < \frac{1+s}{1-s}$$

◆

In the next two examples we look at the two critical cases in the power-like nonlinearity, adding a logarithmic weight.

EXAMPLE 4.9: Lower critical case for powers | We consider here $f(t) = t^{1+2s} \ln^\alpha(1+t)$, $\alpha > 0$. In this case

$$\frac{t f'(t)}{f(t)} = \frac{(1+2s)f(t) + \frac{\alpha t f(t)}{(1+t) \ln(1+t)}}{f(t)} = 1 + 2s + \frac{\alpha t}{(1+t) \ln(1+t)}.$$

Condition (1.23) turns into

$$\int_u^{+\infty} \left(\frac{t}{t^{1+2s} \ln^\alpha(1+t)} \right)^{1/(2s)} dt = \int_u^{+\infty} \frac{dt}{t \ln^{\alpha/(2s)}(1+t)} < +\infty$$

which is fulfilled only for $\alpha > 2s$. Also, hypothesis (1.25) becomes

$$\int_{t_0}^{+\infty} t^{1+2s-2/(1-s)} \ln^\alpha(1+t) dt < +\infty$$

which is satisfied by any $\alpha > 0$ since $(1+2s)(1-s) - 2 < s - 1$. \blacklozenge

EXAMPLE 4.10: Upper critical case for powers | We consider here $f(t) = t^{\frac{1+s}{1-s}} \ln^{-\beta}(1+t)$, $\beta > 0$. In this case

$$\frac{tf'(t)}{f(t)} = \frac{\frac{1+s}{1-s}f(t) - \frac{\beta tf(t)}{(1+t)\ln t}}{f(t)} = \frac{1+s}{1-s} - \frac{\beta t}{(1+t)\ln(1+t)}$$

Hypothesis (1.23) turns into

$$\int_u^{+\infty} \left(\frac{t \ln^\beta(1+t)}{t^{(1+s)/(1-s)}} \right)^{1/(2s)} dt = \int_u^{+\infty} \frac{\ln^{\beta/(2s)}(1+t)}{t^{1/(1-s)}} dt < +\infty$$

which is fulfilled for any $\beta > 0$. Also, hypothesis (1.25) becomes

$$\int_{t_0}^{+\infty} t^{-1} \ln^{-\beta}(1+t) dt < +\infty$$

which is satisfied by any $\beta > 1$. \blacklozenge

4.5 COMMENTS ON HYPOTHESES (1.22) AND (1.25)

The present section will be devoted to the explanation of the difficulties of problem (1.26) when one of hypotheses (1.25) or (1.22) fails.

We recall that hypothesis (1.22) is the one needed to guarantee that the function U defined in (4.7) belongs to $L^1(\Omega)$ (cf. Lemma 4.1), while (1.25) has been used to show that $EU = +\infty$ (cf. Lemma 4.3). These features are essential in proving that U is a supersolution to (1.26). Roughly speaking, condition (1.22) gives a lower growth condition at infinity of the nonlinear term f : in the power case $f(t) = t^p$ it corresponds to $p > 1 + 2s$ (cf. Example 4.8). On the other hand hypothesis (1.25) gives an upper growth condition. Note that in case (1.25) fails, we have two issues: not only the candidate supersolution U does not satisfy $EU = +\infty$, but also the approximate problem (4.15) does not have any solution.

LEMMA 4.11 | *In case (1.25) fails, problem*

$$\begin{cases} (-\Delta)^s u_1 = -f(u_1) & \text{in } \Omega \\ u_1 = 0 & \text{in } \mathcal{C}\Omega \\ Eu_1 = 1 & \text{on } \partial\Omega \end{cases} \quad (4.17)$$

does not admit any weak or pointwise solution.

Proof. In both cases the solution would satisfy $u_1 \geq c\delta^{s-1}$ in Ω , for some $c > 0$. If u_1 was a weak solution then for any $\phi \in \mathcal{T}(\Omega)$

$$\int_{\Omega} u_1 (-\Delta)^s \phi + \int_{\Omega} f(u_1) \phi = \int_{\partial\Omega} D_s \phi$$

where

$$\int_{\Omega} f(u_1) \phi \geq C \int_{\Omega} f(c \delta^{s-1}) \delta^s = +\infty$$

because (1.25) does not hold, a contradiction.

If u_1 was a pointwise solution, then by Lemma 5.4 it would be a weak solution on any subdomain $D \subset \overline{D} \subset \Omega$. Therefore

$$u_1(x) = - \int_D G_D(x, y) f(u_1(y)) dy + \int_{\partial D} P_D(x, y) u_1(y) dy.$$

If u_0 denotes the s -harmonic function induced by $Eu = 1$, then $u_1 \leq u_0$ in Ω and

$$u_1(x) \leq - \int_D G_D(x, y) f(u_1(y)) dy + \int_{\partial D} P_D(x, y) u_0(y) dy = - \int_D G_D(x, y) f(u_1(y)) dy + u_0(x).$$

Fix $x \in \Omega$. Letting now $D \nearrow \Omega$ we have that $G_D(x, y) \uparrow G_{\Omega}(x, y)$ and

$$\int_{\Omega} G_{\Omega}(x, y) f(u_1(y)) dy \geq c \delta(x)^s \int_{\{2\delta(y) < \delta(x)\}} \delta(y)^s f(u_1(y)) dy = +\infty$$

because (1.25) does not hold, a contradiction. □

In this Chapter we would like to point out some elements that may risk to be unclear if left implicit. In the first paragraph we discuss the relation between pointwise solutions and weak L^1 solutions. The second paragraph states in what sense the singular boundary trace is attained by weak solutions, when the data are not smooth. The third and paragraph deals with the definition of weak L^1 solution given by Chen and Véron [26], which amounts to be equivalent to Definition 1.8.

5.1 POINTWISE SOLUTIONS VS. WEAK L^1 SOLUTIONS

LEMMA 5.1 | *Solutions provided by Theorem 1.9 are distributional solutions, meaning that for any $\psi \in C_c^\infty(\Omega)$ it holds*

$$\int_{\Omega} u(-\Delta)^s \psi = \int_{\Omega} \psi d\lambda - \int_{\mathcal{C}\Omega} (-\Delta)^s \psi d\mu.$$

Proof. Consider a sequence $\{\eta_k\}_{k \in \mathbb{N}}$ of bump functions, such that $0 \leq \dots \leq \eta_k \leq \eta_{k+1} \leq \dots \leq 1$ and $\eta_k \uparrow \chi_{\Omega}$. Define $f_k := \eta_k(-\Delta)^s \psi \in C_c^\infty(\Omega)$ and the corresponding $\phi_k \in \mathcal{T}(\Omega)$. Then we know that

$$\int_{\Omega} u f_k = \int_{\Omega} \phi_k d\lambda - \int_{\mathcal{C}\Omega} (-\Delta)^s \phi_k d\mu + \int_{\partial\Omega} D_s \phi_k d\nu.$$

Since $|f_k| \leq |(-\Delta)^s \psi| < C$ in Ω , the left-hand side converges to $\int_{\Omega} u(-\Delta)^s \psi$ by dominated convergence. In the same way $|\phi_k| \leq \int_{\Omega} G_{\Omega}(\cdot, y) |f_k(y)| dy \leq C\delta^s$ and

$$|(-\Delta)^s \phi_k(x)| \leq C_{N,s} \int_{\Omega} \frac{|\phi_k(y)|}{|x-y|^{N+2s}} \leq C \int_{\Omega} \frac{\delta(y)^s}{|x-y|^{N+2s}} \leq C\delta(x)^{-s} \min\{1, \delta(x)^{-N-s}\},$$

thus we have the convergence on the first and the second addend on the left-hand side. Finally, we have that $\phi_k \delta^{-s}$ converges uniformly towards $\psi \delta^{-s}$ in Ω , so that $D_s \phi_k \rightarrow D_s \psi = 0$ as $k \uparrow +\infty$; this, along with

$$|D_s \phi_k(z)| = \left| \int_{\Omega} M_{\Omega}(x, z) f_k(x) dx \right| \leq \int_{\Omega} M_{\Omega}(x, z) |(-\Delta)^s \psi(x)| dx < C(N, \Omega, s)$$

would prove the convergence

$$\int_{\partial\Omega} D_s \phi_k d\nu \xrightarrow{k \uparrow +\infty} \int_{\partial\Omega} D_s \psi d\nu = 0$$

completing the proof of the lemma. Indeed, by (2.26), it holds

$$|\phi_k(x) - \psi(x)| \leq \int_{\Omega} G_{\Omega}(x, y) (1 - \eta_k(y)) |(-\Delta)^s \psi(x)| dy \leq \delta(x)^s \|(-\Delta)^s \psi\|_{L^\infty(\Omega)} \int_{\Omega_k} \frac{dy}{|x-y|^{N-s}}$$

where $\Omega_k := \{\eta_k < 1\} \subset \Omega$ has measure converging to 0, if the sequence $\{\eta_k\}_{k \in \mathbb{N}}$ is properly chosen. \square

LEMMA 5.2 | Suppose $f \in L_{loc}^\infty(\Omega)$, and consider a $u \in L^1(\mathbb{R}^N, (1 + |x|)^{-N-2s} dx)$ satisfying

$$(-\Delta)^s u = f \quad \text{in a distributional sense}$$

i.e. for any $\psi \in C_c^\infty(\Omega)$

$$\int_{\Omega} u(-\Delta)^s \psi = \int_{\Omega} f \psi - \int_{\mathcal{C}\Omega} u(-\Delta)^s \psi.$$

Then $u \in C^\beta(\Omega)$ for any $\beta \in [0, 2s)$. If $f \in C_{loc}^\alpha(\Omega)$ for some $\alpha > 0$ such that $\alpha + 2s \notin \mathbb{N}$, then $u \in C_{loc}^{2s+\alpha}(\Omega)$.

Proof. This can be proved as in [78, Proposition 2.8]. \square

COROLLARY 5.3 | Take $f \in C_{loc}^\alpha(\Omega)$ for some $\alpha > 0$ such that $\alpha + 2s \notin \mathbb{N}$ with

$$\int_{\Omega} |f| \delta^s < +\infty,$$

a Radon measure $\mu \in \mathcal{M}(\mathcal{C}\Omega)$ such that

$$\int_{\mathcal{C}\Omega} \delta^{-s} \min\{1, \delta^{-N-s}\} d|\mu| < +\infty,$$

a finite Radon measure $\nu \in \mathcal{M}(\partial\Omega)$ and $u : \mathbb{R}^N \rightarrow \mathbb{R}$ a weak L^1 solution to

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = \mu & \text{in } \mathcal{C}\Omega \\ Eu = \nu & \text{on } \partial\Omega. \end{cases}$$

Then $(-\Delta)^s u(x) = f(x)$ holds pointwisely for any $x \in \Omega$.

Proof. This is a combination of the previous Lemma with [78, Proposition 2.4]. \square

LEMMA 5.4 | Take $f \in C_{loc}^\alpha(\Omega)$ for some $\alpha \in (0, 1)$, $h \in C(\partial\Omega)$ and $u : \mathbb{R}^N \rightarrow \mathbb{R}$ a pointwise solution to

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = g & \text{in } \mathcal{C}\Omega \\ Eu = h & \text{on } \partial\Omega. \end{cases} \quad (5.1)$$

If

$$\int_{\Omega} |f| \delta^s < +\infty, \quad \int_{\mathcal{C}\Omega} |g| \delta^{-s} \min\{1, \delta^{-N-s}\} < +\infty, \quad h \in C(\partial\Omega)$$

then u is also a weak L^1 solution to the same problem.

Proof. We refer to Theorem 1.9 for the existence and uniqueness of a weak L^1 solution v to problem (5.1). By Corollary 5.3, $v \in C_{loc}^{2s+\alpha}(\Omega)$ and $(-\Delta)^s v = f$ holds pointwisely in Ω . So

$$\begin{cases} (-\Delta)^s(u - v) = 0 & \text{in } \Omega \\ u - v = 0 & \text{in } \mathcal{C}\Omega \\ E(u - v) = 0 & \text{on } \partial\Omega. \end{cases}$$

in a pointwise sense. In particular, $u - v \in C(\Omega)$ since harmonic functions are smooth. Define $\Omega^+ := \{x \in \Omega : u(x) > v(x)\}$: $u - v$ is a nonnegative s -harmonic function and, by [12, Lemma 5 and Theorem 1], it decomposes into the sum of the s -harmonic function induced by the $E_{\Omega^+}(u - v)$ trace and the one by its values on $\mathcal{C}\Omega^+$. But $E_{\Omega^+}(u - v) = 0$ on $\partial\Omega^+$ as it is implied by the singular trace datum in (5.1) and the continuity on $\partial\Omega^+ \cap \Omega$, while $u - v \leq 0$ in $\mathcal{C}\Omega^+$. This yields $\Omega^+ = \emptyset$ and $v \geq u$ in Ω . Repeating the argument we deduce also $u \leq v$ and this completes the proof of the lemma. \square

5.2 VARIATIONAL WEAK SOLUTIONS VS. WEAK L^1 SOLUTIONS

DEFINITION 5.5 | Given $f \in L^\infty(\Omega)$, a variational weak solution of

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega \end{cases} \quad (5.2)$$

is a function $u \in H^s(\mathbb{R}^N)$ such that $u \equiv 0$ in $\mathcal{C}\Omega$ and for any other $v \in H^s(\mathbb{R}^N)$ such that $v \equiv 0$ in $\mathcal{C}\Omega$ it holds

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v = \int_{\Omega} f v.$$

LEMMA 5.6 | $\mathcal{T}(\Omega) \subset H^s(\mathbb{R}^N)$.

Proof. Consider $\phi \in \mathcal{T}(\Omega)$. The fractional Laplacian $(-\Delta)^{s/2} \phi$ is a continuous function decaying like $|x|^{-N-s}$ at infinity. So $\|(-\Delta)^{s/2} \phi\|_{L^2(\mathbb{R}^N)} < \infty$ and we can apply [37, Proposition 3.6]. \square

PROPOSITION 5.7 | Let $f \in L^\infty(\Omega)$. Let u be a variational weak solution of (5.2). Then it is also a weak L^1 solution to the problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega \\ Eu = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Consider $\phi \in \mathcal{T}(\Omega)$:

$$\int_{\Omega} u (-\Delta)^s \phi = \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} \phi = \int_{\Omega} f \phi.$$

where we have used Lemma 5.6 on ϕ . \square

PROPOSITION 5.8 | Let $f \in L^\infty(\Omega)$. Let u be a weak L^1 solution to the problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega \\ Eu = 0 & \text{on } \partial\Omega. \end{cases}$$

Then it is also a variational weak solution of (5.2).

Proof. Call w the variational weak solution of (5.2). By the previous Lemma, w is also a weak L^1 solution. We must conclude $u = w$ by the uniqueness of a weak L^1 solution. \square

5.3 TRACES

With Section 2.4 and Lemma 2.7 we provided some facts on the boundary behaviour of solutions to linear problems with some assumptions on the data. In this section we recover the boundary behaviour in presence of rough data. We use the notion of weak trace introduced by Ponce [71, Proposition 3.5].

PROPOSITION 5.9 | Let $\lambda \in \mathcal{M}(\Omega)$ be a Radon measure such that

$$\int_{\Omega} \delta^s d|\lambda| < +\infty. \quad (5.3)$$

Then the weak solution to

$$\begin{cases} (-\Delta)^s u = \lambda & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega \\ Eu = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies

$$\lim_{\eta \downarrow 0} \frac{1}{\eta} \int_{\{\delta(x) < \eta\}} \delta(x)^{1-s} u(x) dx = 0. \quad (5.4)$$

Proof. By using the Jordan decomposition on $\lambda = \lambda^+ - \lambda^-$, we can suppose that λ is a nonnegative measure without loss of generality. Fix $\sigma \in (0, s)$ and exchange the order of integration in

$$\int_{\{\delta(x) < \eta\}} \delta(x)^{1-s} u(x) dx = \int_{\delta(x) < \eta} \delta(x)^{1-s} \int_{\Omega} G_{\Omega}(x, y) d\lambda(y) dx.$$

Our claim is that

$$\int_{\Omega} G_{\Omega}(x, y) \delta(x)^{1-s} \chi_{(0, \eta)}(\delta(x)) dx \leq \begin{cases} C\eta^{1+\sigma} \delta(y)^{s-\sigma} & \text{if } \delta(y) \geq \eta \\ C\eta \delta(y)^s & \text{if } \delta(y) < \eta. \end{cases} \quad (5.5)$$

This would prove

$$\begin{aligned} \frac{1}{\eta} \int_{\Omega} \left(\int_{\Omega} G_{\Omega}(x, y) \delta(x)^{1-s} \chi_{(0, \eta)}(\delta(x)) dx \right) d\lambda(y) &\leq \\ &\leq C\eta^{\sigma} \int_{\{\delta(y) \geq \eta\} \cap \Omega} \delta(y)^{s-\sigma} d\lambda(y) + C \int_{\{\delta(y) < \eta\} \cap \Omega} \delta(y)^s d\lambda(y) \end{aligned}$$

where the second addend converges to 0 as $\eta \downarrow 0$ by (5.3). For the first addend, we have that $\eta^{\sigma} \delta(y)^{s-\sigma}$ converges pointwisely to zero for any $y \in \Omega$ and $\eta^{\sigma} \delta(y)^{s-\sigma} \leq \delta(y)^s$ if $y \in \Omega \cap \{\delta(y) > \eta\}$, therefore we have the convergence to 0 by dominated convergence.

We turn now to the proof of (5.5). For any $y \in \Omega$ one has

$$\int_{\Omega} G_{\Omega}(x, y) \delta(x)^{1-s} \chi_{(0, \eta)}(\delta(x)) dx \leq \eta^{1+\sigma} \int_{\Omega} G_{\Omega}(x, y) \delta(x)^{-s-\sigma} dx \leq C\eta^{1+\sigma} \delta(y)^{s-\sigma} \quad (5.6)$$

where we have used the regularity at the boundary in (1.16). In particular (5.6) holds when $\delta(y) > \eta$.

To prove the other part of (5.5) we write (dropping from now on multiplicative constants depending on N, Ω and s)

$$\int_{\Omega} G_{\Omega}(x, y) \delta(x)^{1-s} \chi_{(0, \eta)}(\delta(x)) dx \leq \eta^{1-s} \int_{\{\delta(x) < \eta\} \cap \Omega} \frac{(\delta(x) \delta(y) \wedge |x - y|^2)^s}{|x - y|^N} \delta(x)^{1-s} dx$$

and, by exploiting Lemma (8.10) below, we are allowed to perform the computations only in the case where $\partial\Omega$ is locally flat where the above reads as

$$\int_0^{\eta} \int_B \frac{[x_N y_N \wedge (|x' - y'|^2 + |x_N - y_N|^2)]^s}{(|x' - y'|^2 + |x_N - y_N|^2)^{N/2}} \cdot x_N^{1-s} dx' dx_N.$$

where $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and $y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. First note that we can suppose $y' = 0$ without loss of generality and $a \wedge b \leq 2ab/(a+b)$ when $a, b > 0$. With the change of variable $x_N = y_N t$ and $x' = y_N \xi$, we reduce to

$$y_N^{1+s} \int_0^{\eta/y_N} \int_{B_{1/y_N}} \frac{t}{(|\xi|^2 + |t-1|^2)^{N/2-s}} \cdot \frac{d\xi}{(|\xi|^2 + |t-1|^2 + t)^s} dt$$

and, passing to polar coordinates in the ξ variable

$$\begin{aligned} y_N^{1+s} \int_0^{\eta/y_N} \int_0^{1/y_N} \frac{t \rho^{N-2}}{(\rho^2 + |t-1|^2)^{N/2-s}} \cdot \frac{d\rho}{(\rho^2 + |t-1|^2 + t)^s} dt &\leq \\ &\leq y_N^{1+s} \int_0^{\eta/y_N} \int_0^{1/y_N} \frac{t \rho}{(\rho^2 + |t-1|^2)^{3/2-s}} \cdot \frac{d\rho}{(\rho^2 + |t-1|^2 + t)^s} dt. \end{aligned}$$

We deal first with the integral in the ρ variable¹

$$\begin{aligned} & t \int_0^{1/y_N} \frac{\rho}{(\rho^2 + |t-1|^2)^{3/2-s}} \cdot \frac{d\rho}{(\rho^2 + |t-1|^2 + t)^s} \leq \\ & \leq \frac{t}{(|t-1|^2 + t)^s} \int_0^1 \frac{\rho}{(\rho^2 + |t-1|^2)^{3/2-s}} d\rho + t \int_1^{1/y_N} \frac{\rho}{(\rho^2 + |t-1|^2)^{3/2}} d\rho \\ & \leq \frac{t}{(|t-1|^2 + t)^s} \cdot \frac{(\rho^2 + |t-1|^2)^{-1/2+s}}{2s-1} \Big|_{\rho=0}^1 + \frac{t}{(1 + |t-1|^2)^{1/2}}. \end{aligned}$$

We then have

$$\begin{aligned} & t \int_0^{1/y_N} \frac{\rho}{(\rho^2 + |t-1|^2)^{3/2-s}} \cdot \frac{d\rho}{(\rho^2 + |t-1|^2 + t)^s} \leq \\ & \leq \begin{cases} \frac{t}{(|t-1|^2 + t)^s |t-1|^{1-2s}} + \frac{t}{(1 + |t-1|^2)^{1/2}} & s \in (0, 1/2), \\ \frac{t(1 + |t-1|^2)^{s-1/2}}{(|t-1|^2 + t)^s} + \frac{t}{(1 + |t-1|^2)^{1/2}} & s \in (1/2, 1). \end{cases} \end{aligned}$$

The two quantities are both integrable in $t = 1$ and converge to a positive constant as $t \uparrow +\infty$, therefore

$$y_N^{1+s} \int_0^{\eta/y_N} \int_0^{1/y_N} \frac{t \rho^{N-2}}{(\rho^2 + |t-1|^2)^{N/2-s}} \cdot \frac{d\rho}{(\rho^2 + |t-1|^2 + t)^s} dt \leq \eta y_N^s = \eta \delta(y)^s$$

completing the proof of (5.5). \square

PROPOSITION 5.10 | *Let $\mu \in \mathcal{M}(C\Omega)$ be a Radon measure such that*

$$\int_{C\Omega} \delta^{-s} \min\{1, \delta^{-N-s}\} d|\mu| < +\infty. \quad (5.7)$$

Then the s -harmonic function u induced in Ω by μ satisfies

$$\lim_{\eta \downarrow 0} \frac{1}{\eta} \int_{\{\delta(x) < \eta\}} \delta(x)^{1-s} u(x) dx = 0. \quad (5.8)$$

Proof. Again, we can assume without loss of generality that μ is a nonnegative measure. Recall that u is represented by

$$u(x) = - \int_{C\Omega} (-\Delta)^s G_\Omega(x, y) d\mu(y), \quad x \in \Omega$$

and by (1.13)

$$u(x) \leq \int_{C\Omega} \frac{\delta(x)^s}{|x-y|^N} \cdot \frac{d\mu(y)}{\delta(y)^s (1 + \delta(y))^s}.$$

Choose $R > 0$ large enough to have $\Omega \subset\subset B_R$ and split

$$\mu = \mu_1 + \mu_2 := \chi_{B_R} \mu + \chi_{CB_R} \mu :$$

note that

$$\delta(x)^{1-s} \int_{C\Omega} \frac{\delta(x)^s}{|x-y|^N} \cdot \frac{d\mu_2(y)}{\delta(y)^s (1 + \delta(y))^s} \leq \delta(x) \int_{CB_R} \frac{d\mu_2(y)}{\delta(y)^s (1 + \delta(y))^{N+s}}$$

which converges uniformly to 0 as $\delta(x) \downarrow 0$ in view of (5.7). Thus we can suppose that μ is supported in B_R and

$$\frac{1}{\eta} \int_{\{\delta(x) < \eta\}} \delta(x)^{1-s} u(x) dx \leq \int_{C\Omega} \left(\frac{1}{\eta} \int_{\{\delta(x) < \eta\}} \frac{\delta(x)}{|x-y|^N} dx \right) \frac{d\mu(y)}{\delta(y)^s}.$$

¹The computation which follows is not valid in the particular case $s = 1/2$, but with some minor natural modifications the same idea will work.

Once again, we reduce to the flat case as we did in (and with the same notation of) the last proposition. Let $y = (0, -y_N)$ with $y_N > 0$. We have to handle

$$\frac{1}{\eta} \int_0^\eta \int_B x_N [(x_N + y_N)^2 + |x'|^2]^{-N/2} dx' dx_N.$$

With the change of variable $x' = (x_N + y_N)\xi$ we have

$$\frac{1}{\eta} \int_0^\eta \int_B x_N [(x_N + y_N)^2 + |x'|^2]^{-N/2} dx' dx_N = \frac{1}{\eta} \int_0^\eta \frac{x_N}{x_N + y_N} \int_{B_1/(x_N + y_N)} [1 + |\xi|^2]^{-N/2} d\xi dx_N$$

which is uniformly bounded in η for $y_N > 0$. Moreover,

$$\begin{aligned} \frac{1}{\eta} \int_0^\eta \int_B x_N [(x_N + y_N)^2 + |x'|^2]^{-N/2} dx' dx_N &= \int_0^1 \int_B \eta t [(\eta t + y_N)^2 + |x'|^2]^{-N/2} dx' dt \leq \\ &\leq \eta \int_0^1 \int_B [y_N^2 + |x'|^2]^{-N/2} dx' dt \end{aligned}$$

converges to 0 as $\eta \downarrow 0$ for any $y_N > 0$. Therefore, since $\delta^{-s}\mu$ is a finite measure on $\mathcal{C}\Omega$, by dominated convergence the limit

$$\lim_{\eta \downarrow 0} \int_{\mathcal{C}\Omega} \left(\frac{1}{\eta} \int_{\{\delta(x) < \eta\}} \frac{\delta(x)}{|x - y|^N} dx \right) \frac{d\mu(y)}{\delta(y)^s} = 0$$

holds. □

5.4 THE TEST FUNCTION SPACE

In [26] the following definition of weak solution is given.

DEFINITION 5.11 | Given a Radon measure ν such that $\delta^s \in L^1(\Omega, d\nu)$ a function $u \in L^1(\Omega)$ is a weak solution of

$$\begin{cases} (-\Delta)^s u + f(u) = \nu & \text{in } \Omega \\ u = 0 & \text{in } \mathcal{C}\Omega \end{cases}$$

if $f(u) \in L^1(\Omega, \delta^s dx)$

$$\int_\Omega u(-\Delta)^s \xi + \int_\Omega f(u)\xi = \int_\Omega \xi d\nu$$

for any $\xi \in \mathbb{X}_s \subset C(\mathbb{R}^N)$, i.e.

1. $\text{supp} \xi \subseteq \overline{\Omega}$
2. $(-\Delta)^s \xi(x)$ is pointwisely defined for any $x \in \Omega$ and $\|(-\Delta)^s \xi\|_{L^\infty(\Omega)} < +\infty$
3. there exist a positive $\varphi \in L^1(\Omega, \delta^s dx)$ and $\varepsilon_0 > 0$ such that

$$|(-\Delta)_\varepsilon^s \xi(x)| = \left| \int_{\mathcal{C}B_\varepsilon(x)} \frac{\xi(x) - \xi(y)}{|x - y|^{N+2s}} dy \right| \leq \varphi(x) \quad \text{for all } \varepsilon \in (0, \varepsilon_0].$$

The test space \mathbb{X}_s in Definition 5.11 is quite different from the space $\mathcal{T}(\Omega)$ which is used in Definition 1.8. Still, testing a Dirichlet problem against one or the other does not yield two different solutions, i.e. the two notions of weak L^1 solutions are equivalent. We split the proof of this fact into two lemmas.

LEMMA 5.12 | $\mathcal{T}(\Omega) \subset \mathbb{X}_s$.

Proof. Pick $\phi \in \mathcal{T}(\Omega)$. Properties 1. and 2. of Definition 5.11 are satisfied by construction. In order to prove 3. write for $\delta(x) < 2\varepsilon$

$$\begin{aligned} (-\Delta)_\varepsilon^s \phi(x) &= \psi(x) - PV \int_{B_\varepsilon(x)} \frac{\phi(x) - \phi(y)}{|x - y|^{N+2s}} dy = \\ &= \psi(x) - PV \int_{B_{\delta(x)/2}(x)} \frac{\phi(x) - \phi(y)}{|x - y|^{N+2s}} dy - \int_{B_\varepsilon(x) \setminus B_{\delta(x)/2}(x)} \frac{\phi(x) - \phi(y)}{|x - y|^{N+2s}} dy. \end{aligned} \quad (5.9)$$

with $\psi := (-\Delta)^s \phi|_\Omega \in C_c^\infty(\Omega)$. Consider $\alpha \in (0, s)$. For the first integral

$$\begin{aligned} \left| PV \int_{B_{\delta(x)/2}(x)} \frac{\phi(x) - \phi(y)}{|x - y|^{N+2s}} dy \right| &\leq \|\phi\|_{C^{2s+\alpha}(B_{\delta(x)/2}(x))} \int_{B_{\delta(x)/2}(x)} \frac{dy}{|x - y|^{N-\alpha}} \\ &= \|\phi\|_{C^{2s+\alpha}(B_{\delta(x)/2}(x))} \frac{\omega_{N-1}}{\alpha} \left(\frac{\delta(x)}{2} \right)^\alpha \end{aligned}$$

where, by [78, Proposition 2.8]

$$\begin{aligned} \|\phi\|_{C^{2s+\alpha}(B_{\delta(x)/2}(x))} &= 2^{2s+\alpha} \delta(x)^{-2s-\alpha} \left\| \phi \left(x + \frac{\delta(x)}{2} \cdot \right) \right\|_{C^{2s+\alpha}(B)} \leq \\ &\leq C \delta(x)^{-2s-\alpha} \left(\left\| \phi \left(x + \frac{\delta(x)}{2} \cdot \right) \right\|_{L^\infty(B)} + \left\| \psi \left(x + \frac{\delta(x)}{2} \cdot \right) \right\|_{C^\alpha(B)} \right) \\ &\leq C \delta(x)^{-2s-\alpha} \left(\|\phi\|_{L^\infty(B_{\delta(x)/2}(x))} + \delta(x)^\alpha \|\psi\|_{C^\alpha(B_{\delta(x)/2}(x))} \right) \\ &\leq C \delta(x)^{-2s-\alpha} (\|\psi\|_{L^\infty(\mathbb{R}^N)} \delta(x)^s + \delta(x)^\alpha \|\psi\|_{C^\alpha(\mathbb{R}^N)}) \leq C \|\psi\|_{C^\alpha(\mathbb{R}^N)} \delta(x)^{-2s}. \end{aligned}$$

The integration far from x gives

$$\begin{aligned} \left| \int_{B_\varepsilon(x) \setminus B_{\delta(x)/2}(x)} \frac{\phi(x) - \phi(y)}{|x - y|^{N+2s}} dy \right| &\leq \|\phi\|_{C^s(\mathbb{R}^N)} \int_{B_\varepsilon(x) \setminus B_{\delta(x)/2}(x)} \frac{dy}{|x - y|^{N+s}} \leq \\ &\leq \|\phi\|_{C^s(\mathbb{R}^N)} \int_{\mathbb{R}^N \setminus B_{\delta(x)/2}} \frac{dz}{|z|^{N+s}} \leq \|\phi\|_{C^s(\mathbb{R}^N)} \frac{\omega_{N-1}}{s} \left(\frac{2}{\delta(x)} \right)^s. \end{aligned}$$

All this entails

$$\delta(x)^s |(-\Delta)_\varepsilon^s \phi(x)| \leq \delta(x)^s |\psi(x)| + C \|\psi\|_{C^\alpha(\mathbb{R}^N)} \delta(x)^{\alpha-s} + C \|\phi\|_{C^s(\mathbb{R}^N)}, \quad \text{when } \delta(x) < 2\varepsilon.$$

For $\delta(x) \geq 2\varepsilon$ one does not have the second integral on the right-hand side of (5.9) whereas the first one is computed on the ball of radius ε , but the same computations can be run. This proves the statement of the Lemma. \square

LEMMA 5.13 | Given a Radon measure $\nu \in \mathcal{M}(\Omega)$ such that $\delta^s \in L^1(\Omega, d\nu)$, if a function $u \in L^1(\Omega)$ satisfies

$$\int_\Omega u(-\Delta)^s \xi = \int_\Omega \xi d\nu, \quad \text{for any } \xi \in \mathcal{T}(\Omega), \quad (5.10)$$

then the same holds true for any $\xi \in \mathbb{X}_s$.

Proof. Pick $\xi \in \mathbb{X}_s$: by definition, $\zeta := (-\Delta)^s \xi \in L^\infty(\Omega)$. Consider the standard mollifier $\eta \in C_c^\infty(\mathbb{R}^N)$ and $\eta_\varepsilon(x) := \varepsilon^{-N} \eta(x/\varepsilon)$. Then

$$\zeta_\varepsilon := \zeta \chi_\Omega * \eta_\varepsilon \in C^\infty(\mathbb{R}^N) \text{ and } \|\zeta_\varepsilon\|_{L^\infty(\Omega)} \leq \|\zeta\|_{L^\infty(\Omega)}. \quad (5.11)$$

Define ξ_ε as the solution to

$$\begin{cases} (-\Delta)^s \xi_\varepsilon = \zeta_\varepsilon & \text{in } \Omega \\ \xi_\varepsilon = 0 & \text{in } \mathcal{C}\Omega \\ E\xi_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Also, for $\rho > 0$ small consider

$$\Omega_\rho := \{x \in \Omega : \delta(x) > \rho\}$$

and a bump function $b_\rho \in C_c^\infty(\mathbb{R}^N)$ such that

$$b_\rho \equiv 1 \text{ in } \Omega_{2\rho}, \quad b_\rho \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega_\rho, \quad 0 \leq b_\rho \leq 1 \text{ in } \mathbb{R}^N.$$

Then $\zeta_{\varepsilon,\rho} := b_\rho \zeta_\varepsilon \in C_0^\infty(\Omega)$. Let $\xi_{\varepsilon,\rho} \in \mathcal{T}(\Omega)$ be the function induced by $\zeta_{\varepsilon,\rho}$. By (5.10),

$$\int_\Omega u \zeta_{\varepsilon,\rho} = \int_\Omega \xi_{\varepsilon,\rho} d\nu. \quad (5.12)$$

It holds $\zeta_{\varepsilon,\rho} \rightarrow \zeta_\varepsilon$ as $\rho \downarrow 0$, with $\|\zeta_{\varepsilon,\rho}\|_{L^\infty(\Omega)} \leq \|\zeta_\varepsilon\|_{L^\infty(\Omega)}$ and

$$|\xi_{\varepsilon,\rho}(x)| \leq C \|\zeta_{\varepsilon,\rho}\|_{L^\infty(\Omega)} \delta(x)^s \leq \|\zeta_\varepsilon\|_{L^\infty(\Omega)} \delta(x)^s$$

so that we can push equality (5.12) to the limit to deduce, by dominated convergence,

$$\int_\Omega u \zeta_\varepsilon = \int_\Omega \xi_\varepsilon d\nu. \quad (5.13)$$

Similarly, since $\|\zeta_\varepsilon\|_{L^\infty(\Omega)} \leq \|\zeta\|_{L^\infty(\Omega)}$, letting $\varepsilon \downarrow 0$ yields

$$\int_\Omega u \zeta = \int_\Omega \xi d\nu.$$

□

— Part II —

Boundary blow-up solutions for $(-\Delta|_{\Omega})^s$

Introduction and main results

Given a bounded domain Ω of the Euclidean space \mathbb{R}^N , the *spectral fractional Laplacian* operator $(-\Delta|_{\Omega})^s$, $s \in (0, 1)$, is classically defined as a fractional power of the Laplacian with homogeneous Dirichlet boundary conditions, seen as a self-adjoint operator in the Lebesgue space $L^2(\Omega)$, see (6.2) below. This provides a nonlocal operator of elliptic type with *homogeneous* boundary conditions. Recent bibliography on this operator can be found e.g. in Bonforte, Sire and Vázquez [14], Grubb [51], Caffarelli and Stinga [20], Servadei and Valdinoci [77].

One aspect of the theory is however left unanswered: the formulation of natural *nonhomogeneous* boundary conditions. A first attempt can be found in the work of Dhifli, Mâagli and Zribi [36]. The investigations that have resulted in the present Part turn out, we hope, to shed some further light on this question. We provide a weak formulation, which is well-posed in the sense of Hadamard, for linear problems of the form

$$\begin{cases} (-\Delta|_{\Omega})^s u = \mu & \text{in } \Omega, \\ \frac{u}{h_1} = \zeta & \text{on } \partial\Omega, \end{cases} \quad (6.1)$$

where h_1 is a reference function, see (6.8) below, with prescribed singular behaviour at the boundary. Namely, h_1 is bounded above and below by constant multiples of $\delta^{-(2-2s)}$ and the left-hand side of the boundary condition must be understood as a limit as δ converges to zero. In other words, unlike the classical Dirichlet problem for the Laplace operator, nonhomogeneous boundary conditions must be singular. In addition, if the data μ, ζ are smooth, the solution blows up¹ at the fixed rate $\delta^{-(2-2s)}$. In fact, for the special case of positive s -harmonic functions, that is when $\mu = 0$, the singular boundary condition was already identified in previous works emphasizing the probabilistic and potential theoretic aspects of the problem: see e.g. Song and Vondraček [82], Glover, Pop-Stojanovic, Rao, Šikić, Song and Vondraček [47] and Song [81] for the spectral fractional Laplacian.

Turning to nonlinear problems, even more singular boundary conditions arise: in the above system, if $\mu = -u^p$ for suitable values of p , one may choose $\zeta = +\infty$, in the sense that the solution u will blow up at a higher rate with respect to $\delta^{-(2-2s)}$ and controlled by the (scale-invariant) one $\delta^{-2s/(p-1)}$. Note that the value $\zeta = +\infty$ is not admissible for linear problems. This has been already observed in the context of the fractional Laplacian in the previous Part, see [2], and this is what we prove here for the spectral fractional Laplacian. Interestingly, the range of admissible exponents p is different according to which operator one works with.

¹This is very similar indeed to the theory of nonhomogeneous boundary conditions for the fractional Laplacian - although in that case the blow-up rate is of order $\delta^{-(1-s)}$ - as analysed in Part I of this manuscript and from a different perspective by Grubb [50].

6.1 MAIN RESULTS

For clarity, we list here the definitions and statements that we use. First recall the definition of the spectral fractional Laplacian:

DEFINITION 6.1 | Let $\Omega \subset \mathbb{R}^N$ a bounded domain and let $\{\varphi_j\}_{j \in \mathbb{N}}$ be a Hilbert basis of $L^2(\Omega)$ consisting of eigenfunctions of the Dirichlet Laplacian $-\Delta|_\Omega$, associated to the eigenvalues λ_j , $j \in \mathbb{N}$, i.e.² $\varphi_j \in H_0^1(\Omega) \cap C^\infty(\Omega)$ and $-\Delta\varphi_j = \lambda_j\varphi_j$ in Ω . Given $s \in (0, 1)$, consider the Hilbert space³

$$H(2s) := \left\{ v = \sum_{j=1}^{\infty} \widehat{v}_j \varphi_j \in L^2(\Omega) : \|v\|_{H(2s)}^2 = \sum_{j=0}^{\infty} \lambda_j^{2s} |\widehat{v}_j|^2 < \infty \right\}.$$

The spectral fractional Laplacian of $u \in H(2s)$ is the function belonging to $L^2(\Omega)$ given by the formula

$$(-\Delta|_\Omega)^s u = \sum_{j=1}^{\infty} \lambda_j^s \widehat{u}_j \varphi_j. \quad (6.2)$$

Note that $C_c^\infty(\Omega) \subset H(2s) \hookrightarrow L^2(\Omega)$. So, the operator $(-\Delta|_\Omega)^s$ is unbounded, densely defined and with bounded inverse $(-\Delta|_\Omega)^{-s}$ in $L^2(\Omega)$. Alternatively, for almost every $x \in \Omega$,

$$(-\Delta|_\Omega)^s u(x) = PV \int_\Omega [u(x) - u(y)] J(x, y) dy + \kappa(x) u(x), \quad (6.3)$$

where, letting $p_\Omega(t, x, y)$ denote the heat kernel of $-\Delta|_\Omega$,

$$J(x, y) = \frac{s}{\Gamma(1-s)} \int_0^\infty \frac{p_\Omega(t, x, y)}{t^{1+s}} dt \quad \text{and} \quad \kappa(x) = \frac{s}{\Gamma(1-s)} \int_\Omega \left(1 - \int_\Omega p_\Omega(t, x, y) dy \right) \frac{dt}{t^{1+s}} \quad (6.4)$$

are respectively⁴ the *jumping kernel* and the *killing measure*, see Song and Vondraček [82, formulas (3.3) and (3.4)]. For the reader's convenience, we provide a proof of (6.3) in Paragraph 8.3.1. We assume from now on that

$$\Omega \text{ is of class } C^{1,1}.$$

In particular, sharp bounds are known for the heat kernel P_Ω , see (7.9) below, and provide in turn sharp estimates for $J(x, y)$, see (7.13) below, so that the right-hand side of (6.3) remains well-defined for every $x \in \Omega$ under the assumption that $u \in C^{2s+\varepsilon}(\Omega) \cap L^1(\Omega, \delta(x)dx)$ for some $\varepsilon > 0$. This allows us to *define* the spectral fractional Laplacian of functions which *do not* vanish on the boundary of Ω . As a simple example, observe that the function $u(x) \equiv 1$, $x \in \Omega$, does not belong to $H(2s)$ if $s \geq 1/4$, yet it solves (6.1) for $\mu = \kappa$ and $\zeta = 0$.

DEFINITION 6.2 | The Green function and the Poisson kernel of the spectral fractional Laplacian are defined respectively by

$$G_{|\Omega}^s(x, y) = \frac{1}{\Gamma(s)} \int_0^\infty p_\Omega(t, x, y) t^{s-1} dt, \quad x, y \in \Omega, x \neq y, s \in (0, 1], \quad (6.5)$$

where p_Ω denotes the heat kernel of $-\Delta|_\Omega$, and by

$$P_{|\Omega}^s(x, y) := -\frac{\partial}{\partial \nu_y} G_{|\Omega}^s(x, y), \quad x \in \Omega, y \in \partial\Omega. \quad (6.6)$$

where ν is the outward unit normal to $\partial\Omega$.

²See Brezis [15, Theorem 9.31].

³When Ω is smooth, $H(2s)$ coincides with the Sobolev space $H^{2s}(\Omega)$ if $0 < s < 1/4$, $H_{00}^s(\Omega)$ if $s \in \{1/4, 3/4\}$, $H_0^{2s}(\Omega)$ otherwise; see Lions and Magenes [62, Theorems 11.6 and 11.7 pp. 70–72].

⁴in the language of potential theory of killed stochastic processes. Note that the integral in (6.3) must be understood in the sense of principal values. To see this, look at (7.13).

In Section 7.1, we shall prove that $P_{|\Omega}^s$ is well-defined (see Lemma 7.4) and review some useful identities involving the Green function $G_{|\Omega}^s$ and the Poisson kernel $P_{|\Omega}^s$. Now, let us define weak solutions of (6.1).

DEFINITION 6.3 | *Consider the test function space*

$$\mathcal{T}(\Omega) := (-\Delta|_{\Omega})^{-s} C_c^{\infty}(\Omega) \quad (6.7)$$

and the weight

$$h_1(x) = \int_{\partial\Omega} P_{|\Omega}^s(x, y) d\mathcal{H}(y), \quad x \in \Omega. \quad (6.8)$$

Given two Radon measures $\mu \in \mathcal{M}(\Omega)$ and $\zeta \in \mathcal{M}(\partial\Omega)$ with

$$\int_{\Omega} \delta(x) d|\mu|(x) < \infty, \quad |\zeta|(\partial\Omega) < \infty, \quad (6.9)$$

a function $u \in L_{loc}^1(\Omega)$ is a weak solution to

$$\begin{cases} (-\Delta|_{\Omega})^s u = \mu & \text{in } \Omega \\ \frac{u}{h_1} = \zeta & \text{on } \partial\Omega \end{cases} \quad (6.10)$$

if, for any $\psi \in \mathcal{T}(\Omega)$,

$$\int_{\Omega} u (-\Delta|_{\Omega})^s \psi = \int_{\Omega} \psi d\mu - \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu} d\zeta. \quad (6.11)$$

We shall prove that $\mathcal{T}(\Omega) \subseteq C_0^1(\overline{\Omega})$, see Lemma 7.18, so that all integrals above are well-defined. Equation (6.11) is indeed a weak formulation of (6.10), as the following lemma shows.

LEMMA 6.4 | 1. (weak solutions are distributional solutions) Assume that $u \in L_{loc}^1(\Omega)$ is a weak solution of (6.10). Then in fact, $u \in L^1(\Omega, \delta(x)dx)$ and $(-\Delta|_{\Omega})^s u = \mu$ in the sense of distributions i.e. for any $\psi \in C_c^{\infty}(\Omega)$, $\frac{(-\Delta|_{\Omega})^s \psi}{\delta}$ is bounded in Ω and

$$\int_{\Omega} u (-\Delta|_{\Omega})^s \psi = \int_{\Omega} \psi d\mu,$$

moreover the boundary condition holds in the sense that for every $\phi \in C(\overline{\Omega})$

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{\{\delta(x) < t\}} \frac{u(x)}{h_1(x)} \phi(x) d\mathcal{H}(x) = \int_{\partial\Omega} \phi(x) d\zeta(x)$$

whenever $\mu \in \mathcal{M}(\Omega)$ satisfies (6.9) and $\zeta \in L^1(\partial\Omega)$.

2. (for smooth data, weak solutions are classical) Assume that $u \in L_{loc}^1(\Omega)$ is a weak solution of (6.10), where $\mu \in C^{\alpha}(\overline{\Omega})$ for some α such that $\alpha + 2s \notin \mathbb{N}$ and $\zeta \in C(\partial\Omega)$. Then, $(-\Delta|_{\Omega})^s u$ is well-defined by (6.3) for every $x \in \Omega$, $(-\Delta|_{\Omega})^s u(x) = \mu(x)$ for all $x \in \Omega$ and for all $x_0 \in \partial\Omega$,

$$\lim_{\substack{x \rightarrow x_0 \\ x \in \Omega}} \frac{u(x)}{h_1(x)} = \zeta(x_0).$$

3. (classical solutions are weak solutions) Assume that $u \in C^{2s+\varepsilon}(\Omega)$ is such that $u/h_1 \in C(\overline{\Omega})$. Let $\mu = (-\Delta|_{\Omega})^s u$ be given by (6.3) and $\zeta = u/h_1|_{\partial\Omega}$. Then, u is a weak solution of (6.10).

We present some facts about harmonic functions in Section 7.2 with an eye kept on their singular boundary trace in Section 7.3.

We prove the well-posedness of (6.10) in Section 7.4, namely

THEOREM 6.5 | Given two Radon measures $\mu \in \mathcal{M}(\Omega)$ and $\zeta \in \mathcal{M}(\partial\Omega)$ such that (6.9) holds, there exists a unique function $u \in L^1_{loc}(\Omega)$ which is a weak solution to (6.10). Moreover, for a.e. $x \in \Omega$,

$$u(x) = \int_{\Omega} G^s_{|\Omega}(x, y) d\mu(y) + \int_{\partial\Omega} P^s_{|\Omega}(x, y) d\zeta(y) \quad (6.12)$$

and

$$\|u\|_{L^1(\Omega, \delta(x)dx)} \leq C(\Omega, N, s) (\|\delta\mu\|_{\mathcal{M}(\Omega)} + \|\zeta\|_{\mathcal{M}(\partial\Omega)}). \quad (6.13)$$

In addition, the following estimates hold.

$$\|u\|_{L^p(\Omega, \delta(x)dx)} \leq C_1 \|\delta\mu\|_{\mathcal{M}(\Omega)} \quad \text{if } \zeta = 0 \text{ and } p \in \left[1, \frac{N+1}{N+1-2s}\right) \quad (6.14)$$

$$\|u\|_{C^\alpha(\overline{\omega})} \leq C_2 (\|\mu\|_{L^\infty(\omega)} + \|\zeta\|_{\mathcal{M}(\partial\Omega)}) \quad \text{if } \omega \subset\subset \Omega \text{ and } \alpha \in (0, 2s) \quad (6.15)$$

$$\|u\|_{C^{2s+\alpha}(\overline{\omega})} \leq C_3 (\|\mu\|_{C^\alpha(\overline{\omega})} + \|\zeta\|_{\mathcal{M}(\partial\Omega)}) \quad \text{if } \omega \subset\subset \Omega \text{ and } 2s + \alpha \notin \mathbb{N}. \quad (6.16)$$

In the above $C_1 = C_1(\Omega, N, s, p)$, $C_2 = C_2(\Omega, \omega, N, s, \alpha)$, $C_3 = C_3(\Omega, \omega, N, s, \alpha)$.

In Section 8.1 we solve nonlinear Dirichlet problems, by proving

THEOREM 6.6 | Let $g(x, t) : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Carathéodory function and $h : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a nondecreasing function such that $g(x, 0) = 0$ and for a.e. $x \in \Omega$ and all $t > 0$,

$$0 \leq g(x, t) \leq h(t) \quad \text{where} \quad h(\delta^{-(2-2s)})\delta \in L^1(\Omega).$$

Then, problem

$$\begin{cases} (-\Delta|_{\Omega})^s u = -g(x, u) & \text{in } \Omega \\ \frac{u}{h_1} = \zeta & \text{on } \partial\Omega \end{cases} \quad (6.17)$$

has a solution $u \in L^1(\Omega, \delta(x)dx)$ for any $\zeta \in C(\partial\Omega)$, $\zeta \geq 0$. In addition, if $t \mapsto g(x, t)$ is nondecreasing then the solution is unique.

Finally, with Section 8.2 we prove

THEOREM 6.7 | Let

$$p \in \left(1 + s, \frac{1}{1-s}\right).$$

Then, there exists a nonnegative function $u \in L^1(\Omega, \delta(x)dx) \cap C^\infty(\Omega)$ solving

$$\begin{cases} (-\Delta|_{\Omega})^s u = -u^p & \text{in } \Omega, \\ \frac{u}{h_1} = +\infty & \text{on } \partial\Omega \end{cases} \quad (6.18)$$

in the following sense: the first equality holds pointwise and in the sense of distributions, the boundary condition is understood as a pointwise limit. In addition, there exists a constant $C = C(\Omega, N, s, p)$ such that

$$0 \leq u \leq C\delta^{-\frac{2s}{p-1}}.$$

Nonhomogeneous boundary conditions

7.1 GREEN FUNCTION AND POISSON KERNEL

In the following three lemmas¹, we establish some useful identities for the Green function defined by (6.5).

LEMMA 7.1 | ([48, formula (17)]) *Let $f \in L^2(\Omega)$. For almost every $x \in \Omega$, $G_{|\Omega}^s(x, \cdot)f \in L^1(\Omega)$ and*

$$(-\Delta|_{\Omega})^{-s}f(x) = \int_{\Omega} G_{|\Omega}^s(x, y)f(y) dy \quad \text{for a.e. } x \in \Omega.$$

Proof. If φ_j is an eigenfunction of $-\Delta|_{\Omega}$, then

$$\begin{aligned} \int_{\Omega} G_{|\Omega}^s(x, y)\varphi_j(y) dy &= \int_0^{\infty} \frac{t^{s-1}}{\Gamma(s)} \int_{\Omega} p_{\Omega}(t, x, y)\varphi_j(y) dy dt \\ &= \int_0^{\infty} \frac{t^{s-1}}{\Gamma(s)} e^{-\lambda_j t} \varphi_j(x) dt = \frac{\lambda_j^{-s}}{\Gamma(s)} \varphi_j(x) \int_0^{\infty} t^{s-1} e^{-t} dt = \lambda_j^{-s} \varphi_j(x) = (-\Delta|_{\Omega})^{-s} \varphi_j(x). \end{aligned}$$

By linearity, if $f \in L^2(\Omega)$ is a linear combination of eigenvectors $f = \sum_{j=1}^M \widehat{f}_j \varphi_j$, then

$$\int_{\Omega} G_{|\Omega}^s(x, y) \sum_{j=1}^M \widehat{f}_j \varphi_j(y) dy = \sum_{j=1}^M \widehat{f}_j \lambda_j^{-s} \varphi_j(x)$$

and so, letting

$$\mathbb{G}_{|\Omega}^s f := \int_{\Omega} G_{|\Omega}^s(\cdot, y) f(y) dy, \quad (7.1)$$

we have

$$\|\mathbb{G}_{|\Omega}^s f\|_{H(2s)}^2 = \sum_{j=1}^M \lambda_j^{2s} \cdot |\widehat{f}_j|^2 \lambda_j^{-2s} = \|f\|_{L^2(\Omega)}^2. \quad (7.2)$$

Thus the map $\mathbb{G}_{|\Omega}^s : f \mapsto \mathbb{G}_{|\Omega}^s f$ uniquely extends to a linear isometry from $L^2(\Omega)$ to $H(2s)$, which coincides with $(-\Delta|_{\Omega})^{-s}$. It remains to prove that the identity (7.1) remains valid a.e. for $f \in L^2(\Omega)$. By standard parabolic theory, the function $(t, x) \mapsto \int_{\Omega} p_{\Omega}(t, x, y) dy$ is bounded (by 1) and smooth in $[0, T] \times \omega$ for every $T > 0$, $\omega \subset\subset \Omega$. Hence, for every $x \in \Omega$, $G_{|\Omega}^s(x, \cdot) \in L^1(\Omega)$. Assume first that $f = \psi \in C_c^{\infty}(\Omega)$ and take a sequence $\{\psi_k\}_{k \in \mathbb{N}}$ in the linear span of the eigenvectors $\{\varphi_j\}_{j \in \mathbb{N}}$ such that $\{\psi_k\}_{k \in \mathbb{N}}$ converges to ψ in $L^2(\Omega)$. The convergence is in fact uniform and so (7.1) holds for $f = \psi$. Indeed, by standard elliptic regularity, there exist constants $C = C(N, \Omega)$, $k = k(N)$ such that any eigenfunction satisfies

$$\|\nabla \varphi_j\|_{L^{\infty}(\Omega)} \leq (C\lambda_j)^k \|\varphi_j\|_{L^2(\Omega)} = (C\lambda_j)^k.$$

¹which hold even if the domain Ω is not $C^{1,1}$

In particular, taking C larger if needed,

$$\left\| \frac{\varphi_j}{\delta} \right\|_{L^\infty(\Omega)} \leq (C\lambda_j)^k. \quad (7.3)$$

Now write $\psi = \sum_{j=1}^{\infty} \widehat{\psi}_j \varphi_j$ and fix $m \in \mathbb{N}$. Integrating by parts m times yields

$$\widehat{\psi}_j = \int_{\Omega} \psi \varphi_j = -\frac{1}{\lambda_j} \int_{\Omega} \psi \Delta \varphi_j = -\frac{1}{\lambda_j} \int_{\Omega} \Delta \psi \varphi_j = \dots = \frac{(-1)^m}{\lambda_j^m} \int_{\Omega} \Delta^m \psi \varphi_j$$

which implies that

$$|\widehat{\psi}_j| \leq \frac{\|\Delta^m \psi\|_{L^2(\Omega)}}{\lambda_j^m}, \quad (7.4)$$

i.e. the spectral coefficients of ψ converge to 0 faster than any polynomial. This and (7.3) imply that $\{\psi_k\}_{k \in \mathbb{N}}$ converges to ψ uniformly, as claimed.

Take at last $f \in L^2$ and a sequence $\{f_k\}_{k \in \mathbb{N}}$ in $C_c^\infty(\Omega)$ of nonnegative functions such that $\{f_k\}_{k \in \mathbb{N}}$ converges to $|f|$ a.e. and in $L^2(\Omega)$. By (7.2), $\|G_{|\Omega}^s f_k\|_{L^2} \leq \|f_k\|_{L^2}$ and by Fatou's lemma, we deduce that $G_{|\Omega}^s(x, \cdot) f \in L^1(\Omega)$ for a.e. $x \in \Omega$ and the desired identity follows. \square

LEMMA 7.2 | ([48, formula (8)]) *For a.e. $x, y \in \Omega$,*

$$\int_{\Omega} G_{|\Omega}^{1-s}(x, \xi) G_{|\Omega}^s(\xi, y) d\xi = G_{|\Omega}^1(x, y). \quad (7.5)$$

Proof. Clearly, given an eigenfunction φ_j ,

$$(-\Delta|_{\Omega})^{-s} (-\Delta|_{\Omega})^{s-1} \varphi_j = \lambda_j^{-s} \lambda_j^{s-1} \varphi_j = (-\Delta|_{\Omega})^{-1} \varphi_j$$

so $(-\Delta|_{\Omega})^{-s} \circ (-\Delta|_{\Omega})^{s-1} = (-\Delta|_{\Omega})^{-1}$ in $L^2(\Omega)$. By the previous lemma and Fubini's theorem, we deduce that for $\varphi \in L^2(\Omega)$ and a.e. $x \in \Omega$,

$$\int_{\Omega^2} G_{|\Omega}^{1-s}(x, \xi) G_{|\Omega}^s(\xi, y) \varphi(y) d\xi dy = \int_{\Omega} G_{|\Omega}^1(x, y) \varphi(y) dy$$

and so (7.5) holds almost everywhere. \square

LEMMA 7.3 | *For any $\psi \in C_c^\infty(\Omega)$,*

$$(-\Delta|_{\Omega})^s \psi = (-\Delta) \circ (-\Delta|_{\Omega})^{s-1} \psi = (-\Delta|_{\Omega})^{s-1} \circ (-\Delta) \psi. \quad (7.6)$$

Proof. The identity clearly holds if ψ is an eigenfunction. If $\psi \in C_c^\infty(\Omega)$, its spectral coefficients have fast decay and the result follows by writing the spectral decomposition of ψ . Indeed, thanks to (7.4) and (7.3), we may easily work by density to establish (7.6). \square

Let us turn to the definition and properties of the Poisson kernel. Recall that, for $x \in \Omega, y \in \partial\Omega$, the Poisson kernel of the Dirichlet Laplacian is given by

$$P_{|\Omega}^1(x, y) = -\frac{\partial}{\partial \nu_y} G_{|\Omega}^1(x, y).$$

LEMMA 7.4 | *The function*

$$P_{|\Omega}^s(x, y) := -\frac{\partial}{\partial \nu_y} G_{|\Omega}^s(x, y)$$

is well-defined for $x \in \Omega, y \in \partial\Omega$ and $P_{|\Omega}^s(x, \cdot) \in C(\partial\Omega)$ for any $x \in \Omega$. Furthermore, there exists a constant $C > 0$ depending on N, s, Ω only such that

$$\frac{1}{C} \frac{\delta(x)}{|x - y|^{N+2-2s}} \leq P_{|\Omega}^s(x, y) \leq C \frac{\delta(x)}{|x - y|^{N+2-2s}} \quad (7.7)$$

and

$$\int_{\Omega} G_{|\Omega}^{1-s}(x, \xi) P_{|\Omega}^s(\xi, y) d\xi = P_{|\Omega}^1(x, y). \quad (7.8)$$

REMARK 7.5 | When Ω is merely Lipschitz, one must work with the Martin kernel in place of the Poisson kernel, see [47].

Proof. of Lemma 7.4. Take $x, z \in \Omega$, $y \in \partial\Omega$. Then,

$$\frac{G_{|\Omega}^s(x, z)}{\delta(z)} = \frac{1}{\Gamma(s)\delta(z)} \int_0^\infty p_\Omega(t, x, z) t^{s-1} dt = \frac{|z-x|^{2s}}{\Gamma(s)\delta(z)} \int_0^\infty p_\Omega(|z-x|^2\tau, x, z) \tau^{s-1} d\tau.$$

Since Ω has $C^{1,1}$ boundary, given $x \in \Omega$, $p_\Omega(\cdot, x, \cdot) \in C^1((0, +\infty) \times \overline{\Omega})$ and the following heat kernel bound holds (cf. Davies, Simon and Zhang [32, 33, 88]):

$$\left[\frac{\delta(x)\delta(y)}{t} \wedge 1 \right] \frac{1}{c_1 t^{N/2}} e^{-|x-y|^2/(c_2 t)} \leq p_\Omega(t, x, y) \leq \left[\frac{\delta(x)\delta(y)}{t} \wedge 1 \right] \frac{c_1}{t^{N/2}} e^{-c_2|x-y|^2/t}, \quad (7.9)$$

where c_1, c_2 are constants depending on Ω, N only. So,

$$\frac{|z-x|^{2s}}{\delta(z)} p_\Omega(|z-x|^2\tau, x, z) \tau^{s-1} \leq C|z-x|^{2s-N-2} \delta(x) \tau^{s-2-N/2} e^{-c_2\tau} \quad (7.10)$$

and the reverse inequality

$$\frac{1}{C}|z-x|^{2s-N-2} \delta(x) \tau^{s-2-N/2} e^{-1/(c_2\tau)} \leq \frac{|z-x|^{2s}}{\delta(z)} p_\Omega(|z-x|^2\tau, x, z) \tau^{s-1}$$

also holds for $\tau \geq \delta(x)\delta(z)|z-x|^{-2}$. As $z \rightarrow y \in \partial\Omega$, the right-hand-side of (7.10) obviously converges in $L^1(0, +\infty, d\tau)$ so we may apply the generalized dominated convergence theorem to deduce that $P_{|\Omega}^s(x, y)$ is well-defined, satisfies (7.7) and

$$P_{|\Omega}^s(x, y) = -\frac{\partial}{\partial\nu_y} G_{|\Omega}^s(x, y) = \lim_{z \rightarrow y} \frac{G_{|\Omega}^s(x, z)}{\delta(z)} = -\frac{1}{\Gamma(s)} \int_0^\infty \frac{\partial}{\partial\nu_y} p_\Omega(t, x, y) t^{s-1} dt.$$

From this last formula we deduce also that, for any fixed $x \in \Omega$, the function $P_{|\Omega}^s(x, \cdot) \in C(\partial\Omega)$: indeed, having chosen a sequence $\{y_k\}_{k \in \mathbb{N}} \subset \partial\Omega$ converging to some $y \in \partial\Omega$, we have

$$\left| P_{|\Omega}^s(x, y_k) - P_{|\Omega}^s(x, y) \right| \leq \frac{1}{\Gamma(s)} \int_0^\infty \left| \frac{\partial}{\partial\nu_y} p_\Omega(t, x, y_k) - \frac{\partial}{\partial\nu_y} p_\Omega(t, x, y) \right| t^{s-1} dt$$

where, by (7.9)

$$\left| \frac{\partial}{\partial\nu_y} p_\Omega(t, x, y) \right| \leq \frac{c_1 \delta(x)}{t^{N/2+1}} e^{-c_2|x-y|^2/t} \leq \frac{c_1 \delta(x)}{t^{N/2+1}} e^{-c_2\delta(x)^2/t} \quad \text{for any } y \in \partial\Omega,$$

so that $|P_{|\Omega}^s(x, y_k) - P_{|\Omega}^s(x, y)| \rightarrow 0$ as $k \uparrow \infty$ by dominated convergence.

By similar arguments, $G_{|\Omega}^s$ is a continuous function on $\overline{\Omega^2} \setminus \{(x, y) : x = y\}$. And so, by (7.5), we have

$$-\frac{\partial}{\partial\nu_y} \int_\Omega G_{|\Omega}^{1-s}(x, \xi) G_{|\Omega}^s(\xi, y) d\xi = P_{|\Omega}^1(x, y).$$

Let us compute the derivative of the left-hand side alternatively. We have

$$\int_\Omega G_{|\Omega}^{1-s}(x, \xi) \frac{G_{|\Omega}^s(\xi, z)}{\delta(z)} d\xi = \int_{\mathbb{R}^+ \times \Omega} f(t, \xi, z) dt d\xi,$$

where, having fixed $x \in \Omega$,

$$f(t, \xi, z) = \frac{G_{|\Omega}^{1-s}(x, \xi)}{\Gamma(s)\delta(z)} p_\Omega(t, \xi, z) t^{s-1} \leq C|x-\xi|^{2s-N} \left[\frac{\delta(x)\delta(\xi)}{|x-\xi|^2} \wedge 1 \right] t^{s-2-N/2} \delta(\xi) e^{-c_2|z-\xi|^2/t}.$$

For a fixed $\varepsilon > 0$, and $\xi \in \Omega \setminus B(y, \varepsilon)$, $z \in B(y, \varepsilon/2)$, we deduce that

$$f(t, \xi, z) \leq C|x-\xi|^{2s-N} t^{s-2-N/2} e^{-c_\varepsilon/t} \in L^1((0, +\infty) \times \Omega)$$

Similarly, if $t > \varepsilon$,

$$f(t, \xi, z) \leq C|x - \xi|^{2s-N} t^{s-2-N/2} \in L^1((\varepsilon, +\infty) \times \Omega).$$

Now,

$$\int_0^\varepsilon t^{s-1-N-2} e^{-c_2 \frac{|\xi-z|^2}{t}} dt \leq |\xi - z|^{2s-N} \int_0^{+\infty} \tau^{s-1-N/2} e^{-c_2/\tau} d\tau.$$

Hence, there exists a constant $C > 0$ independent of ε such that

$$\int_{(0,\varepsilon) \times B(y,\varepsilon)} f(t, \xi, z) dt d\xi \leq C\varepsilon^{2+2s}.$$

It follows from the above estimates and dominated convergence that

$$P_{|\Omega}^1(x, y) = \lim_{z \rightarrow y} \int_{(0,+\infty) \times \Omega} f(t, \xi, z) dt d\xi = \int_{\Omega} G_{|\Omega}^{1-s}(x, \xi) P_{|\Omega}^s(\xi, y) d\xi \quad (7.11)$$

i.e. (7.8) holds. \square

REMARK 7.6 | Thanks to the heat kernel bound (7.9), the following estimate also holds.

$$\frac{1}{C} \frac{1}{|x - y|^{N-2s}} \left(1 \wedge \frac{\delta(x) \delta(y)}{|x - y|^2} \right) \leq G_{|\Omega}^s(x, y) \leq \frac{C}{|x - y|^{N-2s}} \left(1 \wedge \frac{\delta(x) \delta(y)}{|x - y|^2} \right) \quad (7.12)$$

for some constant $C = C(\Omega, N, s)$. Also observe for computational convenience that

$$\frac{1}{2} \left(1 \wedge \frac{\delta(x) \delta(y)}{|x - y|^2} \right) \leq \frac{\delta(x) \delta(y)}{\delta(x) \delta(y) + |x - y|^2} \leq \left(1 \wedge \frac{\delta(x) \delta(y)}{|x - y|^2} \right).$$

7.2 HARMONIC FUNCTIONS AND INTERIOR REGULARITY

By the heat kernel bound (7.9), there exists $C = C(\Omega, N, s) > 0$ such that

$$\frac{1}{C|x - y|^{N+2s}} \left[\frac{\delta(x) \delta(y)}{|x - y|^2} \wedge 1 \right] \leq J(x, y) \leq \frac{C}{|x - y|^{N+2s}} \left[\frac{\delta(x) \delta(y)}{|x - y|^2} \wedge 1 \right]. \quad (7.13)$$

DEFINITION 7.7 | A function $h \in L^1(\Omega, \delta(x) dx)$ is s -harmonic in Ω if for any $\psi \in C_c^\infty(\Omega)$ there holds

$$\int_{\Omega} h (-\Delta|_{\Omega})^s \psi = 0.$$

The above definition makes sense thanks to the following lemma.

LEMMA 7.8 | For any $\psi \in C_c^\infty(\Omega)$, $(-\Delta|_{\Omega})^s \psi \in C_0^1(\overline{\Omega})$ and there exists a constant $C = C(s, N, \Omega, \psi) > 0$ such that

$$|(-\Delta|_{\Omega})^s \psi| \leq C\delta \quad \text{in } \Omega. \quad (7.14)$$

In addition, if $\psi \geq 0$, $\psi \neq 0$, then

$$(-\Delta|_{\Omega})^s \psi \leq -C\delta \quad \text{in } \Omega \setminus \text{supp } \psi. \quad (7.15)$$

Proof. Thanks to (7.3) and (7.4), $(-\Delta|_{\Omega})^s \psi \in C_0^1(\overline{\Omega})$ and

$$\left| \frac{(-\Delta|_{\Omega})^s \psi}{\delta} \right| \leq \sum_{j=1}^{\infty} \lambda_j^s |\widehat{\psi}_j| \left\| \frac{\varphi_j}{\delta} \right\|_{L^\infty(\Omega)} < \infty$$

and (7.14) follows. Let us turn to the case where $\psi \geq 0$, $\psi \not\equiv 0$. We apply formula (6.3) and assume that $x \in \Omega \setminus \text{supp} \psi$. Denote by x^* a point of maximum of ψ and let $2r = \text{dist}(x^*, \text{supp} \psi)$. Then for $y \in B_r(x^*)$, it holds $\psi(y), \delta(y) \geq c_1 > 0$, $r \leq |x - y| \leq c_2$ and so, applying (7.13),

$$\begin{aligned} (-\Delta|_{\Omega})^s \psi(x) &= - \int_{\Omega} \psi(y) J(x, y) dy \\ &\leq -C \int_{B_r(x^*)} \frac{\psi(y)}{|x - y|^{N+2s}} \left[\frac{\delta(x)\delta(y)}{|x - y|^2} \wedge 1 \right] dy \\ &\leq -C \cdot \int_{B_r(x^*)} \frac{c_1}{c_2^{N+2s}} \left[\frac{\delta(x)c_1}{c_2^2} \wedge 1 \right] dy \\ &\leq -\tilde{C} \delta(x). \end{aligned}$$

□

LEMMA 7.9 | *The function $P_{|\Omega}^s(\cdot, z) \in L^1(\Omega, \delta(x)dx)$ is s -harmonic in Ω for any fixed $z \in \partial\Omega$.*

Proof. Thanks to (7.7), $P_{|\Omega}^s(\cdot, z) \in L^1(\Omega, \delta(x)dx)$. Pick $\psi \in C_c^\infty(\Omega)$ and exploit (7.6):

$$\int_{\Omega} P_{|\Omega}^s(\cdot, z) (-\Delta|_{\Omega})^s \psi = \int_{\Omega} P_{|\Omega}^s(\cdot, z) (-\Delta|_{\Omega})^{s-1} [-\Delta \psi].$$

Applying Lemma 7.1, the Fubini's Theorem and (7.8), the above quantity is equal to

$$\int_{\Omega} P_{|\Omega}^1(\cdot, z) (-\Delta) \psi = 0.$$

□

LEMMA 7.10 | *For any finite Radon measure $\zeta \in \mathcal{M}(\partial\Omega)$, let*

$$h(x) = \int_{\partial\Omega} P_{|\Omega}^s(x, z) d\zeta(z), \quad x \in \Omega. \quad (7.16)$$

Then, h is s -harmonic in Ω . In addition, there exists a constant $C = C(N, s, \Omega) > 0$ such that

$$\|h\|_{L^1(\Omega, \delta(x)dx)} \leq C \|\zeta\|_{\mathcal{M}(\partial\Omega)}. \quad (7.17)$$

Conversely, for any s -harmonic function $h \in L^1(\Omega, \delta(x)dx)$, $h \geq 0$, there exists a finite Radon measure $\zeta \in \mathcal{M}(\partial\Omega)$, $\zeta \geq 0$, such that (7.16) holds.

Proof. Since $P_{|\Omega}^s(x, \cdot)$ is continuous, h is well-defined. By (7.7),

$$\delta(x)|h(x)| \leq C \int_{\partial\Omega} \frac{d|\zeta|(z)}{|x - z|^{N-2s}}$$

so that $h \in L^1(\Omega, \delta(x)dx)$ and (7.17) holds. Pick now $\psi \in C_c^\infty(\Omega)$:

$$\int_{\Omega} h(x) (-\Delta|_{\Omega})^s \psi(x) dx = \int_{\partial\Omega} \left(\int_{\Omega} P_{|\Omega}^s(x, z) (-\Delta|_{\Omega})^s \psi(x) dx \right) d\zeta(z) = 0$$

in view of Lemma 7.9. Conversely, let h denote a nonnegative s -harmonic function. By Definition 7.7 and by equation (7.6), we have for any $\psi \in C_c^\infty(\Omega)$

$$\begin{aligned} 0 &= \int_{\Omega} h(x) (-\Delta|_{\Omega})^s \psi(x) dx = \int_{\Omega} h(x) (-\Delta|_{\Omega})^{s-1} \circ (-\Delta) \psi(x) dx = \\ &= \int_{\Omega} \left(\int_{\Omega} G_{|\Omega}^{1-s}(x, \xi) h(\xi) d\xi \right) (-\Delta) \psi(x) dx, \end{aligned}$$

so that $\int_{\Omega} G_{|\Omega}^{1-s}(x, \xi) h(\xi) d\xi$ is a (standard) nonnegative harmonic function. In particular (cf. e.g. [5, Corollary 6.15]), there exists a finite Radon measure $\zeta \in \mathcal{M}(\partial\Omega)$ such that $\int_{\Omega} G_{|\Omega}^{1-s}(x, \xi) h(\xi) d\xi = \int_{\partial\Omega} P_{|\Omega}^1(x, y) d\zeta(y) \in C^\infty(\Omega)$. We now exploit equation (7.8) to deduce that

$$\int_{\Omega} G_{|\Omega}^{1-s}(x, \xi) \left[h(\xi) - \int_{\partial\Omega} P_{|\Omega}^s(\xi, y) d\zeta(y) \right] d\xi = 0.$$

Since

$$\int_{\Omega} \varphi_1(x) \left(\int_{\Omega} G_{|\Omega}^{1-s}(x, \xi) h(\xi) d\xi \right) dx = \int_{\Omega} h(-\Delta|_{\Omega})^{s-1} \varphi_1 = \frac{1}{\lambda_1^{1-s}} \int_{\Omega} h \varphi_1 < \infty,$$

it holds $\int_{\Omega} G_{|\Omega}^{1-s}(x, \xi) h(\xi) d\xi \in L^1(\Omega, \delta(x) dx)$. Thanks to (7.12), we are allowed to let $G_{|\Omega}^s$ act on it. By (7.5), this leads to

$$\int_{\Omega} G_{|\Omega}^1(x, \xi) \left[h(\xi) - \int_{\partial\Omega} P_{|\Omega}^s(\xi, y) d\zeta(y) \right] d\xi = 0.$$

Take at last $\psi \in C_c^\infty(\Omega)$ and $\varphi = (-\Delta|_{\Omega})^{-1} \psi$. Then,

$$0 = \int_{\Omega} \varphi(x) \left[\int_{\Omega} G_{|\Omega}^1(x, \xi) \left[h(\xi) - \int_{\partial\Omega} P_{|\Omega}^s(\xi, y) d\zeta(y) \right] d\xi \right] dx = \int_{\Omega} \psi(\xi) \left[h(\xi) - \int_{\partial\Omega} P_{|\Omega}^s(\xi, y) d\zeta(y) \right] d\xi$$

and so (7.16) holds a.e. and in fact everywhere thanks to Lemma 7.11 below. \square

LEMMA 7.11 | Take $\alpha > 0$ such that $2s + \alpha \notin \mathbb{N}$ and $f \in C_{loc}^\alpha(\Omega)$. If $u \in L^1(\Omega, \delta(x) dx)$ solves

$$(-\Delta|_{\Omega})^s u = f \quad \text{in } \mathcal{D}'(\Omega),$$

then $u \in C_{loc}^{2s+\alpha}(\Omega)$, the above equation holds pointwise, and given any compact sets $K \subset\subset K' \subset\subset \Omega$, there exists a constant $C = C(s, N, \alpha, K, K', \Omega)$ such that

$$\|u\|_{C^{2s+\alpha}(K)} \leq C \left(\|f\|_{C^\alpha(K')} + \|u\|_{L^1(\Omega, \delta(x) dx)} \right).$$

Similarly, if $f \in L_{loc}^\infty(\Omega)$ and $\alpha \in (0, 2s)$,

$$\|u\|_{C^\alpha(K)} \leq C \left(\|f\|_{L^\infty(K')} + \|u\|_{L^1(\Omega, \delta(x) dx)} \right).$$

In particular, if h is s -harmonic, then $h \in C^\infty(\Omega)$ and the equality $(-\Delta|_{\Omega})^s h(x) = 0$ holds at every point $x \in \Omega$.

Proof. We only prove the former inequality, the proof of the latter follows mutatis mutandis. Given $x \in \Omega$, let

$$v(x) = \int_{\Omega} G_{|\Omega}^{1-s}(x, y) u(y) dy.$$

Observe that v is well-defined and

$$\|v\|_{L^1(\Omega, \delta(x) dx)} \leq C(\Omega, N, s) \|u\|_{L^1(\Omega, \delta(x) dx)} \quad (7.18)$$

Indeed, letting $\varphi_1 > 0$ denote an eigenvector associated to the principal eigenvalue of the Laplace operator, it follows from the Fubini's theorem and Lemma 7.1 that

$$\int_{\Omega} \varphi_1(x) \int_{\Omega} G_{|\Omega}^{1-s}(x, y) |u(y)| dy dx = \lambda_1^{s-1} \int_{\Omega} |u(y)| \varphi_1(y) dy.$$

In addition, $-\Delta v = f$ in $\mathcal{D}'(\Omega)$, since for $\varphi \in C_c^\infty(\Omega)$,

$$\int_{\Omega} v(-\Delta)\varphi = \int_{\Omega} u(-\Delta|_{\Omega})^s \varphi,$$

thanks to the Fubini's theorem, equation (7.6), Lemma 7.1 and Definition 7.7. Observe now that if $\varphi \in C_c^\infty(\Omega)$, then

$$(-\Delta|_\Omega)^{1-s}\varphi = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} t^{s-1} \left(\frac{\varphi - e^{-t\Delta|_\Omega}\varphi}{t} \right) dt.$$

The above identity is straightforward if φ is an eigenfunction and remains true for $\varphi \in C_c^\infty(\Omega)$ by density, using the fast decay of spectral coefficients, see (7.4). So,

$$\begin{aligned} \int_\Omega u \varphi \, dx &= \int_\Omega v (-\Delta|_\Omega)^{1-s}\varphi \, dx = \\ &= \frac{s}{\Gamma(1-s)} \int_\Omega \int_0^\infty v t^{s-1} \left(\frac{\varphi - e^{-t\Delta|_\Omega}\varphi}{t} \right) dt \, dx = \\ &= \frac{s}{\Gamma(1-s)} \int_\Omega \int_0^\infty \varphi t^{s-1} \left(\frac{v - e^{-t\Delta|_\Omega}v}{t} \right) dt \, dx \end{aligned}$$

and

$$u = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} t^{s-1} \left(\frac{v - e^{-t\Delta|_\Omega}v}{t} \right) dt.$$

Choose $\bar{f} \in C_c^\alpha(\mathbb{R}^N)$ such that $\bar{f} = f$ in K' , $\|\bar{f}\|_{C^\alpha(\mathbb{R}^N)} \leq C\|f\|_{C^\alpha(K')}$ and let

$$\bar{u}(x) = c_{N,s} \int_{\mathbb{R}^N} |x-y|^{-(N-2s)} \bar{f}(y) \, dy$$

solve $(-\Delta)^s \bar{u} = \bar{f}$ in \mathbb{R}^N . It is well-known (see e.g. [78]) that $\|\bar{u}\|_{C^{2s+\alpha}(\mathbb{R}^N)} \leq C\|\bar{f}\|_{C^\alpha(\mathbb{R}^N)}$, for a constant C depending only on s, α, N and the measure of the support of \bar{f} . It remains to estimate $u - \bar{u}$. Letting

$$\bar{v}(x) = c_{N,1-s} \int_{\mathbb{R}^N} |x-y|^{-(N-2(1-s))} \bar{u}(y) \, dy,$$

we have as previously that $-\Delta \bar{v} = \bar{f}$ and

$$\bar{u} = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} t^{s-1} \left(\frac{\bar{v} - e^{-t\Delta} \bar{v}}{t} \right) dt,$$

where this time $e^{-t\Delta} \bar{v}(x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \bar{v}(y) \, dy$. Hence,

$$\frac{\Gamma(1-s)}{s} (u - \bar{u}) = \int_0^{+\infty} t^{s-1} \left(\frac{(v - \bar{v}) - e^{-t\Delta}(v - \bar{v})}{t} \right) dt + \int_0^{+\infty} t^{s-1} \left(\frac{e^{-t\Delta|_\Omega}v - e^{-t\Delta} \bar{v}}{t} \right) dt.$$

Fix a compact set K'' such that $K \subset \subset K'' \subset \subset K'$. Since $v - \bar{v}$ is harmonic in K' ,

$$\|v - \bar{v}\|_{C^{2s+\alpha+2}(K'')} \leq C\|v - \bar{v}\|_{L^1(K')} \leq C\|u\|_{L^1(\Omega, \delta(x)dx)}$$

By parabolic regularity,

$$\left\| \int_0^{+\infty} t^{s-1} \left(\frac{(v - \bar{v}) - e^{-t\Delta}(v - \bar{v})}{t} \right) dt \right\|_{C^{2s+\alpha}(K)} \leq C\|u\|_{L^1(\Omega, \delta(x)dx)}.$$

In addition, the function $w = e^{-t\Delta|_\Omega}v - e^{-t\Delta} \bar{v}$ solves the heat equation inside Ω with initial condition $w(0, \cdot) = v - \bar{v}$. Since $\Delta(v - \bar{v}) = 0$ in K' , it follows from parabolic regularity again that $w(t, x)/t$ remains bounded in $C^{2s+\alpha}(K)$ as $t \rightarrow 0^+$, so that again

$$\left\| \int_0^{+\infty} t^{s-1} \left(\frac{e^{-t\Delta|_\Omega}v - e^{-t\Delta} \bar{v}}{t} \right) dt \right\|_{C^{2s+\alpha}(K)} \leq C\|u\|_{L^1(\Omega, \delta(x)dx)}.$$

□

LEMMA 7.12 | Take $\alpha > 0$, $\alpha \notin \mathbb{N}$, and $u \in C_{loc}^{2s+\alpha}(\Omega) \cap L^1(\Omega, \delta(x)dx)$. Given any compact set $K \subset \subset K' \subset \subset \Omega$, there exists a constant $C = C(s, N, \alpha, K, K', \Omega)$ such that

$$\|(-\Delta|_{\Omega})^s u\|_{C^\alpha(K)} \leq C (\|u\|_{C^{2s+\alpha}(K')} + \|u\|_{L^1(\Omega, \delta(x)dx)}).$$

Proof. With a slight abuse of notation, we write

$$(-\Delta|_{\Omega})^{s-1} u(x) = \int_{\Omega} G_{|\Omega}^{1-s}(x, y) u(y) dy.$$

By Lemma 7.11 we have

$$\begin{aligned} \|(-\Delta|_{\Omega})^{s-1} u\|_{C^{2+\alpha}(K)} &\leq C \left(\|u\|_{C^{2s+\alpha}(K')} + \|(-\Delta|_{\Omega})^{s-1} u\|_{L^1(\Omega, \delta(x)dx)} \right) \leq \\ &\leq C (\|u\|_{C^{2s+\alpha}(K')} + \|u\|_{L^1(\Omega, \delta(x)dx)}). \end{aligned}$$

Obviously it holds also

$$\|(-\Delta) \circ (-\Delta|_{\Omega})^{s-1} u\|_{C^\alpha(K)} \leq \|(-\Delta|_{\Omega})^{s-1} u\|_{C^{2+\alpha}(K)}.$$

By (7.6),

$$(-\Delta) \circ (-\Delta|_{\Omega})^{s-1} u = (-\Delta|_{\Omega})^s u \quad \text{in } \mathcal{D}'(\Omega),$$

which concludes the proof. \square

PROPOSITION 7.13 | Let $f \in L^1(\Omega, \delta(x)dx)$ and $u \in L_{loc}^1(\Omega)$. The function

$$u(x) = \int_{\Omega} G_{|\Omega}^s(x, y) f(y) dy$$

belongs to $L^p(\Omega, \delta(x)dx)$ for any $p \in \left[1, \frac{N+1}{N+1-2s}\right)$.

Proof. We start by applying the Jensen's Inequality,

$$\left| \int_{\Omega} G_{|\Omega}^s(x, y) f(y) dy \right|^p \leq \|f\|_{L^1(\Omega, \delta(x)dx)}^{p-1} \int_{\Omega} \left| \frac{G_{|\Omega}^s(x, y)}{\delta(y)} \right|^p \delta(y) f(y) dy,$$

so that

$$\int_{\Omega} |u(x)|^p \delta(x) dx \leq \|f\|_{L^1(\Omega, \delta(x)dx)}^p \sup_{y \in \Omega} \int_{\Omega} \left| \frac{G_{|\Omega}^s(x, y)}{\delta(y)} \right|^p \delta(x) dx$$

and by (7.12) we have to estimate

$$\sup_{y \in \Omega} \int_{\Omega} \frac{1}{|x-y|^{(N-2s)p}} \cdot \frac{\delta(x)^{p+1}}{[|x-y|^2 + \delta(x)\delta(y)]^p} dx$$

Pick $\varepsilon > 0$. Clearly,

$$\sup_{\{y : \delta(y) \geq \varepsilon\}} \int_{\Omega} \frac{1}{|x-y|^{(N-2s)p}} \cdot \frac{\delta(x)^{p+1}}{[|x-y|^2 + \delta(x)\delta(y)]^p} dx \leq C_\varepsilon.$$

Thanks to Lemma 8.10, we may now reduce to the case where the boundary is flat, i.e. when in a neighbourhood A of a given point $y \in \Omega$ such that $\delta(y) < \varepsilon$, there holds $A \cap \partial\Omega \subseteq \{y_N = 0\}$ and $A \cap \Omega \subseteq \{y_N > 0\}$. Without loss of generality, we assume that $y = (0, y_N)$ and $x = (x', x_N) \in B \times (0, 1) \subseteq \mathbb{R}^{N-1} \times \mathbb{R}$. We are left with proving that

$$\int_B dx' \int_0^1 dx_N \frac{1}{[|x'|^2 + |x_N - y_N|^2]^{(N-2s)p/2}} \cdot \frac{x_N^{p+1}}{[|x'|^2 + |x_N - y_N|^2 + x_N y_N]^p}$$

is a bounded quantity. Make the change of variables $x_N = y_N t$ and pass to polar coordinates in x' , with $|x'| = y_N \rho$. Then, the above integral becomes

$$y_N^{-(N+1-2s)p+N+1} \int_0^{1/y_N} d\rho \int_0^{1/y_N} dt \frac{\rho^{N-2}}{[\rho^2 + |t-1|^2]^{(N-2s)p/2}} \cdot \frac{t^{p+1}}{[\rho^2 + |t-1|^2 + t]^p}. \quad (7.19)$$

Now, we split the integral in the t variable into $\int_0^{1/2} + \int_{1/2}^{3/2} + \int_{3/2}^{1/y_N}$. Note that the exponent $-(N+1-2s)p+N+1$ is positive for $p < (N+1)/(N+1-2s)$. We drop multiplicative constants in the computations that follow. The first integral is bounded above by a constant multiple of

$$\int_0^{1/y_N} d\rho \int_0^{1/2} dt \frac{\rho^{N-2}}{[\rho^2 + 1]^{(N-2s)p/2}} \cdot \frac{t^{p+1}}{[\rho^2 + 1 + t]^p} \lesssim \int_0^{1/y_N} d\rho \frac{\rho^{N-2}}{[\rho^2 + 1]^{(N+2-2s)p/2}}$$

which remains bounded as $y_N \downarrow 0$ since

$$p \geq 1 > \frac{N-1}{N+2-2s} \quad \text{implies} \quad (N+2-2s)p - N + 2 > 1.$$

The second integral is of the order of

$$\begin{aligned} & \int_0^{1/y_N} d\rho \int_{1/2}^{3/2} dt \frac{\rho^{N-2}}{[\rho^2 + |t-1|^2]^{(N-2s)p/2}} \cdot \frac{1}{[\rho^2 + 1]^p} = \\ &= \int_0^{1/y_N} d\rho \int_0^{1/2} dt \frac{\rho^{N-2}}{[\rho^2 + t^2]^{(N-2s)p/2}} \cdot \frac{1}{[\rho^2 + 1]^p} \\ &= \int_0^{1/y_N} d\rho \frac{\rho^{N-1-(N-2s)p}}{[\rho^2 + 1]^p} \int_0^{1/(2\rho)} dt \frac{1}{[1 + t^2]^{(N-2s)p/2}} \\ &\lesssim \int_0^\infty d\rho \frac{\rho^{N-1-(N-2s)p}}{[\rho^2 + 1]^p} \end{aligned}$$

which is finite since $p < (N+1)/(N+1-2s) < N/(N-2s)$ implies $N-1-(N-2s)p > -1$ and $p \geq 1 > N/(N+2-2s)$ implies $2p - N + 1 + (N-2s)p > 1$.

We are left with the third integral which is controlled by

$$\begin{aligned} & \int_0^{1/y_N} d\rho \rho^{N-2} \int_{3/2}^{1/y_N} dt \frac{t^{p+1}}{[\rho^2 + t^2]^{(N+2-2s)p/2}} \leq \\ & \leq \int_0^{1/y_N} d\rho \rho^{N-(N+1-2s)p} \int_{3/(2\rho)}^{1/(y_N \rho)} dt \frac{t^{p+1}}{[1 + t^2]^{(N+2-2s)p/2}} \\ & \lesssim \int_0^{1/y_N} d\rho \rho^{N-(N+1-2s)p}. \end{aligned}$$

The exponent $N-(N+1-2s)p > -1$ since $p < (N+1)/(N+1-2s)$, so this third integral is bounded above by a constant multiple of $y_N^{-1-N+(N+1-2s)p}$ which simplifies with the factor in front of (7.19). \square

7.3 BOUNDARY BEHAVIOUR

We first provide the boundary behaviour of the reference function h_1 . Afterwards, in Proposition 7.15 below, we will deal with the weighted trace left on the boundary by harmonic functions induced by continuous boundary data.

LEMMA 7.14 | *Let h_1 be given by (6.8). There exists a constant $C = C(N, \Omega, s) > 0$ such that*

$$\frac{1}{C} \delta^{-(2-2s)} \leq h_1 \leq C \delta^{-(2-2s)}. \quad (7.20)$$

Proof. Restrict without loss of generality to the case where x lies in a neighbourhood of $\partial\Omega$. Take $x^* \in \partial\Omega$ such that $|x - x^*| = \delta(x)$, which exists by compactness of $\partial\Omega$. Take $\Gamma \subset \partial\Omega$ a neighbourhood of x^* in the topology of $\partial\Omega$. By Lemma 8.10 in the Appendix, we can think of $\Gamma \subset \{x_N = 0\}$, $x^* = 0$ and $x = (0, \delta(x)) \in \mathbb{R}^{N-1} \times \mathbb{R}$ without loss of generality. in such a way that it is possible to compute

$$\int_{\Gamma} \frac{\delta(x)}{|x - z|^{N+2-2s}} d\mathcal{H}(z) \asymp \int_{B_r} \frac{\delta(x)}{[|z'|^2 + \delta(x)^2]^{N/2+1-s}} dz'.$$

Recalling (7.7), we have reduced the estimate to

$$\begin{aligned} & \int_{\partial\Omega} P_{|\Omega}^s(x, z) d\mathcal{H}(z) \asymp \\ & \asymp \int_{B_r} \frac{\delta(x)}{[|z'|^2 + \delta(x)^2]^{N/2+1-s}} dz' \asymp \int_0^r \frac{\delta(x) t^{N-2}}{[t^2 + \delta(x)^2]^{N/2+1-s}} dt \\ & = \int_0^{r/\delta(x)} \frac{\delta(x)^N t^{N-2}}{[\delta(x)^2 t^2 + \delta(x)^2]^{N/2+1-s}} dt \asymp \delta(x)^{2s-2} \int_0^{r/\delta(x)} \frac{t^{N-2} dt}{[t^2 + 1]^{N/2+1-s}} \end{aligned}$$

and this concludes the proof, since

$$\int_0^{r/\delta(x)} \frac{t^{N-2} dt}{[t^2 + 1]^{N/2+1-s}} \asymp 1.$$

□

In the following we will use the notation

$$\mathbb{P}_{|\Omega}^s g := \int_{\partial\Omega} P_{|\Omega}^s(\cdot, \theta) g(\theta) d\mathcal{H}(\theta)$$

where σ denotes the Hausdorff measure on $\partial\Omega$, whenever $g \in L^1(\Omega)$.

PROPOSITION 7.15 | *Let $\zeta \in C(\partial\Omega)$. Then, for any $z \in \partial\Omega$,*

$$\frac{\mathbb{P}_{|\Omega}^s \zeta(x)}{h_1(x)} \xrightarrow[x \rightarrow z]{x \in \Omega} \zeta(z) \quad \text{uniformly on } \partial\Omega. \quad (7.21)$$

Proof. Let us write

$$\begin{aligned} \left| \frac{\mathbb{P}_{|\Omega}^s \zeta(x)}{h_1(x)} - \zeta(z) \right| &= \left| \frac{1}{h_1(x)} \int_{\partial\Omega} P_{|\Omega}^s(x, \theta) \zeta(\theta) d\mathcal{H}(\theta) - \frac{h_1(x) \zeta(z)}{h_1(x)} \right| \leq \\ &\leq \frac{1}{h_1(x)} \int_{\partial\Omega} P_{|\Omega}^s(x, \theta) |\zeta(\theta) - \zeta(z)| d\mathcal{H}(\theta) \leq C\delta(x)^{3-2s} \int_{\partial\Omega} \frac{|\zeta(\theta) - \zeta(z)|}{|x - \theta|^{N+2-2s}} d\mathcal{H}(\theta) \leq \\ &\leq C\delta(x) \int_{\partial\Omega} \frac{|\zeta(\theta) - \zeta(z)|}{|x - \theta|^N} d\mathcal{H}(\theta). \end{aligned}$$

It suffices now to repeat the computations in Lemma 2.7 to show that the obtained quantity converges to 0 as $x \rightarrow z$. □

With an approximation argument started from the last Proposition, we can deal with a $\zeta \in L^1(\partial\Omega)$ datum.

THEOREM 7.16 | *For any $\zeta \in L^1(\partial\Omega)$ and any $\phi \in C^0(\overline{\Omega})$ it holds*

$$\frac{1}{t} \int_{\{\delta(x) \leq t\}} \frac{\mathbb{P}_{|\Omega}^s \zeta(x)}{h_1(x)} \phi(x) dx \xrightarrow[t \downarrow 0]{} \int_{\partial\Omega} \phi(y) \zeta(y) d\mathcal{H}(y).$$

Proof. For a general $\zeta \in L^1(\partial\Omega)$, consider a sequence $\{\zeta_k\}_{k \in \mathbb{N}} \subset C(\partial\Omega)$ such that

$$\int_{\partial\Omega} |\zeta_k(y) - \zeta(y)| d\mathcal{H}(y) \xrightarrow{k \uparrow \infty} 0. \quad (7.22)$$

For any fixed $k \in \mathbb{N}$, we have

$$\left| \frac{1}{t} \int_{\{\delta(x) < t\}} \frac{\mathbb{P}_{|\Omega}^s \zeta(x)}{h_1(x)} \phi(x) dx - \int_{\partial\Omega} \phi(x) \zeta(x) d\mathcal{H}(x) \right| \leq \left| \frac{1}{t} \int_{\{\delta(x) < t\}} \frac{\mathbb{P}_{|\Omega}^s \zeta(x) - \mathbb{P}_{|\Omega}^s \zeta_k(x)}{h_1(x)} \phi(x) dx \right| \quad (7.23)$$

$$+ \left| \frac{1}{t} \int_{\{\delta(x) < t\}} \frac{\mathbb{P}_{|\Omega}^s \zeta_k(x)}{h_1(x)} \phi(x) dx - \int_{\partial\Omega} \phi(x) \zeta_k(x) d\mathcal{H}(x) \right| \quad (7.24)$$

$$+ \left| \int_{\partial\Omega} \phi(x) \zeta_k(x) d\mathcal{H}(x) - \int_{\partial\Omega} \phi(x) \zeta(x) d\mathcal{H}(x) \right|. \quad (7.25)$$

Call $\lambda_k := \zeta_k - \zeta$: the term (7.23) equals

$$\frac{1}{t} \int_{\{\delta(x) < t\}} \frac{\mathbb{P}_{|\Omega}^s \lambda_k(x)}{h_1(x)} \phi(x) d\mathcal{H}(x) = \int_{\partial\Omega} \left(\frac{1}{t} \int_{\{\delta(x) < t\}} \frac{P_{|\Omega}^s(x, y)}{h_1(x)} \phi(x) dx \right) \lambda_k(y) d\mathcal{H}(y).$$

Call

$$\Phi(t, y) := \frac{1}{t} \int_{\{\delta(x) < t\}} \frac{P_{|\Omega}^s(x, y)}{h_1(x)} \phi(x) dx.$$

Combining equations (7.7), (7.20) and the boundedness of ϕ , we can prove that Φ is uniformly bounded in t and y . Indeed,

$$|\Phi(t, y)| \leq \frac{\|\phi\|_{L^\infty(\Omega)}}{t} \int_{\{\delta(x) < t\}} \frac{\delta(x)^{3-2s}}{|x-y|^{N+2-2s}} dx \leq \frac{\|\phi\|_{L^\infty(\Omega)}}{t} \int_{\{\delta(x) < t\}} \frac{\delta(x)}{|x-y|^N} dx$$

and reducing our attention to the flat case (see Lemma 8.10 in the Appendix for the complete justification) we estimate (the ' superscript denotes an object living in \mathbb{R}^{N-1})

$$\frac{1}{t} \int_0^t \int_{B'} \frac{x_N}{[|x'|^2 + x_N^2]^{N/2}} dx' dx_N \leq \frac{1}{t} \int_0^t \int_{B'_{1/x_N}} \frac{d\xi}{[|\xi|^2 + 1]^{N/2}} dx_N \leq \int_{\mathbb{R}^{N-1}} \frac{d\xi}{[|\xi|^2 + 1]^{N/2}}.$$

Thus $\int_{\partial\Omega} \Phi(t, y) \lambda_k(y) d\mathcal{H}(y)$ is arbitrarily small in k in view of (7.22).

The term (7.24) converges to 0 as $t \downarrow 0$ because the convergence

$$\frac{\mathbb{P}_{|\Omega}^s \zeta_k(x)}{h_1(x)} \phi(x) \xrightarrow[x \rightarrow z]{x \in \Omega} \zeta_k(z) \phi(z)$$

is uniform in $z \in \partial\Omega$ in view of Proposition 7.15.

Finally, the term (7.25) is arbitrarily small with $k \uparrow +\infty$, because of (7.22). This concludes the proof of the theorem, because

$$\begin{aligned} \lim_{t \downarrow 0} \left| \frac{1}{t} \int_{\{\delta(x) < t\}} \frac{\mathbb{P}_{|\Omega}^s \zeta(x)}{h_1(x)} \phi(x) dx - \int_{\partial\Omega} \phi(x) \zeta(x) d\mathcal{H}(x) \right| &\leq \\ &\leq \|\Phi\|_{L^\infty((0, t_0) \times \partial\Omega)} \int_{\partial\Omega} |\zeta_k(y) - \zeta(y)| d\mathcal{H}(y) + \|\phi\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |\zeta_k(y) - \zeta(y)| d\mathcal{H}(y) \end{aligned}$$

and letting $k \uparrow +\infty$ we deduce the thesis as a consequence of (7.22). \square

Moreover we have also

THEOREM 7.17 | For any $\mu \in \mathcal{M}(\Omega)$, such that

$$\int_{\Omega} \delta d|\mu| < \infty, \quad (7.26)$$

and any $\phi \in C^0(\overline{\Omega})$ it holds

$$\frac{1}{t} \int_{\{\delta(x) \leq t\}} \frac{\mathbb{G}_{|\Omega}^s \mu(x)}{h_1(x)} \phi(x) dx \xrightarrow{t \downarrow 0} 0. \quad (7.27)$$

Proof. By using the Jordan decomposition of $\mu = \mu^+ - \mu^-$ into its positive and negative part, we can suppose without loss of generality that $\mu \geq 0$. Fix some $s' \in (0, s \wedge 1/2)$. Exchanging the order of integration we claim that

$$\int_{\{\delta(x) \leq t\}} G_{|\Omega}^s(x, y) \delta(x)^{2-2s} dx \leq \begin{cases} C t^{2-2s'} \delta(y)^{2s'} & \delta(y) \geq t \\ C t \delta(y) & \delta(y) < t \end{cases} \quad (7.28)$$

where $C = C(N, \Omega, s)$ and does not depend on t , which yields

$$\frac{1}{t} \int_{\Omega} \left(\int_{\{\delta(x) < t\}} G_{|\Omega}^s(x, y) \frac{dx}{h_1(x)} \right) d\mu(y) \leq C t^{1-2s'} \int_{\{\delta(y) \geq t\}} \delta(y)^{2s'} d\mu(y) + C \int_{\{\delta(x) < t\}} \delta(y) d\mu(y).$$

The second addend converges to 0 as $t \downarrow 0$ by (7.26). Since $t^{1-2s'} \delta(y)^{2s'}$ converges pointwisely to 0 in Ω as $t \downarrow 0$ and $t^{1-2s'} \delta(y)^{2s'} \leq \delta(y)$ in $\{\delta(y) \geq t\}$, then the first addend converges to 0 by dominated convergence. This suffices to deduce our thesis (7.27).

Let us turn now to the proof of the claimed estimate (7.28). For the first part we refer to [36, Proposition 7] to say

$$\int_{\{\delta(x) \leq t\}} G_{|\Omega}^s(x, y) \delta(x)^{2-2s} dx \leq t^{2-2s'} \int_{\{\delta(x) \leq t\}} G_{|\Omega}^s(x, y) \delta(x)^{-2s+2s'} dx \leq C t^{2-2s'} \delta^{2s'}.$$

We focus our attention on the case where $\partial\Omega$ is locally flat, i.e. we suppose that in a neighbourhood A of y it holds $A \cap \partial\Omega \subseteq \{x_N = 0\}$ (see Lemma 8.10 in the Appendix to reduce the general case to this one). So, since $\delta(x) = x_N$ and retrieving estimate (7.12) on the Green function, we are dealing with (the ' superscript denotes objects that live in \mathbb{R}^{N-1})

$$\int_0^t \int_{B'(y')} \frac{y_N x_N^{3-2s}}{|x' - y'|^2 + (x_N - y_N)^2 + x_N y_N} \cdot \frac{dx'}{[|x' - y'|^2 + (x_N - y_N)^2]^{(N-2s)/2}} dx_N.$$

From now on we drop multiplicative constants depending only on N and s . Suppose without loss of generality $y' = 0$. Set $x_N = y_N \eta$ and switch to polar coordinates in the x' variable:

$$\int_0^{t/y_N} \int_0^1 \frac{y_N^{5-2s} \eta^{3-2s}}{r^2 + y_N^2 (\eta - 1)^2 + \eta y_N^2} \cdot \frac{r^{N-2} dr}{[r^2 + y_N^2 (\eta - 1)^2]^{(N-2s)/2}} d\eta$$

and then set $r = y_N \rho$ to get

$$\begin{aligned} y_N^2 \int_0^{t/y_N} \eta^{3-2s} \int_0^{1/y_N} \frac{\rho^{N-2}}{[\rho^2 + (\eta - 1)^2]^{(N-2s)/2}} \cdot \frac{d\rho}{\rho^2 + (\eta - 1)^2 + \eta} d\eta &\leq \\ &\leq y_N^2 \int_0^{t/y_N} \eta^{3-2s} \int_0^{1/y_N} \frac{\rho}{[\rho^2 + (\eta - 1)^2]^{(3-2s)/2}} \cdot \frac{d\rho}{\rho^2 + (\eta - 1)^2 + \eta} d\eta. \end{aligned}$$

Consider now $s \in (1/2, 1)$. The integral in the ρ variable is less than

$$\int_0^{1/y_N} \frac{\rho}{[\rho^2 + (\eta - 1)^2]^{(5-2s)/2}} d\rho \leq |\eta - 1|^{-3+2s}$$

so that, integrating in the η variable,

$$y_N^2 \int_0^{t/y_N} \eta^{3-2s} |\eta - 1|^{-3+2s} d\eta \leq t y_N$$

and we prove (7.28) in the case $s \in (1/2, 1)$. Now we study the case $s \in (0, 1/2]$. Split the integration in the η variable into \int_0^2 and \int_2^{1/y_N} : the latter can be treated in the same way as above. For the other one we exploit the inequality $\rho^2 + (\eta - 1)^2 + \eta \geq \frac{3}{4}$ to deduce²:

$$\begin{aligned} y_N^2 \int_0^2 \eta^{3-2s} \left(\int_0^{1/y_N} \frac{\rho}{[\rho^2 + (\eta - 1)^2 + \eta]^{(3-2s)/2}} \cdot \frac{d\rho}{\rho^2 + (\eta - 1)^2} \right) d\eta &\leq \\ &\leq y_N^2 \int_0^2 \eta^{3-2s} \left(\int_0^{1/y_N} \frac{\rho}{[\rho^2 + (\eta - 1)^2]^{(3-2s)/2}} d\rho \right) d\eta \leq y_N^2 \int_0^2 \frac{\eta^{3-2s}}{|\eta - 1|^{1-2s}} d\eta \leq y_N^2. \end{aligned}$$

Note now that, in our set of assumptions, $y_N = \delta(y) < t$. So $y_N^2 \leq t y_N$ and we get to the desired conclusion (7.28) also in the case $s \in (0, 1/2]$. \square

7.4 THE DIRICHLET PROBLEM

Recall the definition of test functions (6.7).

LEMMA 7.18 | $\mathcal{T}(\Omega) \subseteq C_0^1(\overline{\Omega}) \cap C^\infty(\Omega)$. Moreover, for any $\psi \in \mathcal{T}(\Omega)$ and $z \in \partial\Omega$,

$$-\frac{\partial\psi}{\partial\nu}(z) = \int_{\Omega} P_{|\Omega}^s(y, z) (-\Delta|_{\Omega})^s \psi(y) dy. \quad (7.29)$$

Proof. Take $\psi \in \mathcal{T}(\Omega)$ and let $f = (-\Delta|_{\Omega})^s \psi$. Since $f \in C_c^\infty(\Omega)$, the spectral coefficients of f have fast decay (see (7.4)) and so the same holds true for ψ . It follows that $\psi \in C_0^1(\overline{\Omega})$ and $\mathcal{T}(\Omega) \subseteq C_0^1(\overline{\Omega})$. By Lemma 7.1, for all $x \in \overline{\Omega}$,

$$\psi(x) = \int_{\Omega} G_{|\Omega}^s(x, y) f(y) dy$$

Using Lemma 7.4, (7.10) and the dominated convergence theorem, (7.29) follows.

Since $(-\Delta|_{\Omega})^s$ is self-adjoint in $H(2s)$, we know that the equality $(-\Delta|_{\Omega})^s \psi = f$ holds in $\mathcal{D}'(\Omega)$ and the interior regularity follows from Lemma 7.11. \square

LEMMA 7.19: MAXIMUM PRINCIPLE FOR CLASSICAL SOLUTIONS | Consider $u \in C^{2s+\varepsilon}(\Omega) \cap L^1(\Omega, \delta(x) dx)$ such that

$$(-\Delta|_{\Omega})^s u \geq 0 \quad \text{in } \Omega, \quad \liminf_{x \rightarrow \partial\Omega} u(x) \geq 0.$$

Then $u \geq 0$ in Ω . In particular this holds when $u \in \mathcal{T}(\Omega)$.

Proof. Suppose $x^* \in \Omega$ such that $u(x^*) = \min_{\Omega} u < 0$. Then

$$(-\Delta|_{\Omega})^s u(x^*) = \int_{\Omega} [u(x^*) - u(y)] J(x, y) dy + \kappa(x^*) u(x^*) < 0,$$

a contradiction. \square

LEMMA 7.20: MAXIMUM PRINCIPLE FOR WEAK SOLUTIONS | Let $\mu \in \mathcal{M}(\Omega), \zeta \in \mathcal{M}(\partial\Omega)$ be two Radon measures satisfying (6.9) with $\mu \geq 0$ and $\zeta \geq 0$. Consider $u \in L_{loc}^1(\Omega)$ a weak solution to the Dirichlet problem (6.10). Then $u \geq 0$ a.e. in Ω .

Proof. Take $f \in C_c^\infty(\Omega)$, $f \geq 0$ and $\psi = (-\Delta|_{\Omega})^{-s} f \in \mathcal{T}(\Omega)$. By Lemma 7.19, $\psi \geq 0$ in Ω and by Lemma 7.18 $-\frac{\partial\psi}{\partial\nu} \geq 0$ on $\partial\Omega$. Thus, by (6.11), $\int_{\Omega} u f \geq 0$. Since this is true for every $f \in C_c^\infty(\Omega)$, the result follows. \square

²In the computation that follows, in the particular case $s = \frac{1}{2}$ the term $|1 - \eta|^{2s-1}$ must be replaced by $-\ln|1 - \eta|$, but this is harmless.

7.4.1 Proof of Theorem 6.5

Uniqueness is a direct consequence of the Maximum Principle, Lemma 7.20. Let us prove that formula (6.12) defines the desired weak solution. Observe that if u is given by (6.12), then $u \in L^1(\Omega, \delta(x)dx)$. Indeed,

$$\begin{aligned} \int_{\Omega} \left| \varphi_1(x) \int_{\Omega} G_{|\Omega}^s(x, y) d\mu(y) \right| dx &\leq \int_{\Omega} \int_{\Omega} G_{|\Omega}^s(x, y) \varphi_1(x) dx d|\mu|(y) \\ &= \frac{1}{\lambda_1^s} \int_{\Omega} \varphi_1(y) d|\mu|(y) \leq C \|\delta\mu\|_{\mathcal{M}(\Omega)}. \end{aligned} \quad (7.30)$$

This, along with Lemma 7.10, proves that $u \in L^1(\Omega, \delta(x)dx)$ and (6.13). Now, pick $\psi \in \mathcal{T}(\Omega)$ and compute, via the Fubini's Theorem, Lemma 7.1 and Lemma 7.18,

$$\begin{aligned} \int_{\Omega} u(x) (-\Delta|_{\Omega})^s \psi(x) dx &= \\ &= \int_{\Omega} \left(\int_{\Omega} G_{|\Omega}^s(x, y) d\mu(y) \right) (-\Delta|_{\Omega})^s \psi(x) dx + \int_{\Omega} \left(\int_{\partial\Omega} P_{|\Omega}^s(x, z) d\zeta(z) \right) (-\Delta|_{\Omega})^s \psi(x) dx = \\ &= \int_{\Omega} \left(\int_{\Omega} G_{|\Omega}^s(x, y) (-\Delta|_{\Omega})^s \psi(x) dx \right) d\mu(y) + \int_{\partial\Omega} \left(\int_{\Omega} P_{|\Omega}^s(x, z) (-\Delta|_{\Omega})^s \psi(x) dx \right) d\zeta(z) = \\ &= \int_{\Omega} \psi(y) d\mu(y) - \int_{\partial\Omega} \frac{\partial \psi}{\partial \nu}(z) d\zeta(z). \end{aligned}$$

□

7.4.2 Proof of Lemma 6.4

Proof of 1. Consider a sequence $\{\eta_k\}_{k \in \mathbb{N}} \subset C_c^\infty(\Omega)$ of bump functions such that $0 \leq \eta_1 \leq \dots \leq \eta_k \leq \eta_{k+1} \leq \dots \leq 1$ and $\eta_k(x) \uparrow \chi_{\Omega}(x)$ as $k \uparrow \infty$. Consider $\psi \in C_c^\infty(\Omega)$ and define $f_k := \eta_k (-\Delta|_{\Omega})^s \psi \in C_c^\infty(\Omega)$, $\psi_k := (-\Delta|_{\Omega})^{-s} f_k \in \mathcal{T}(\Omega)$.

Let us first note that the integral

$$\int_{\Omega} u (-\Delta|_{\Omega})^s \psi$$

makes sense in view of (7.14) and (6.13). The sequence $\{f_k\}_{k \in \mathbb{N}}$ trivially converges a.e. to $(-\Delta|_{\Omega})^s \psi$, while

$$|\psi_k(x) - \psi(x)| \leq \int_{\Omega} G_{|\Omega}^s(x, y) |(-\Delta|_{\Omega})^s \psi(y)| \cdot (1 - \eta_k(y)) dy$$

converges to 0 for any $x \in \Omega$ by dominated convergence. Since u is a weak solution, it holds

$$\int_{\Omega} u f_k = \int_{\Omega} \psi_k d\mu - \int_{\partial\Omega} \frac{\partial \psi_k}{\partial \nu} d\zeta.$$

By dominated convergence $\int_{\Omega} u f_k \rightarrow \int_{\Omega} u (-\Delta|_{\Omega})^s \psi$ and $\int_{\Omega} \psi_k d\mu \rightarrow \int_{\Omega} \psi d\mu$. Indeed for any $k \in \mathbb{N}$,

$$|f_k| \leq |(-\Delta|_{\Omega})^s \psi| \quad \text{and} \quad |\psi_k| \leq \frac{1}{\lambda_1^s} \left\| \frac{(-\Delta|_{\Omega})^s \psi}{\varphi_1} \right\|_{L^\infty(\Omega)} \varphi_1,$$

where the latter inequality follows from the maximum principle or the representation formula $\psi_k(x) = \int_{\Omega} G_{|\Omega}^s(x, y) f_k(y) dy$. Finally, the convergence

$$\int_{\partial\Omega} \frac{\partial \psi_k}{\partial \nu} d\zeta \xrightarrow{k \uparrow \infty} 0$$

holds by dominated convergence, since

$$\int_{\partial\Omega} \frac{\partial \psi_k}{\partial \nu} d\zeta = \int_{\Omega} \mathbb{P}_{|\Omega}^s \zeta f_k$$

by the Fubini's Theorem, and $\mathbb{P}_\Omega^s \zeta \in L^1(\Omega, \delta(x)dx)$ while $|f_k| \leq |(-\Delta|_\Omega)^s \psi| \leq C\delta$ by Lemma 7.8, so that

$$\int_\Omega \mathbb{P}_\Omega^s \zeta f_k \xrightarrow{k \uparrow \infty} \int_\Omega \mathbb{P}_\Omega^s \zeta f = 0$$

because $\mathbb{P}_\Omega^s \zeta$ is s -harmonic and $f \in C_c^\infty(\Omega)$.

The proof of the boundary trace can be found in Theorems 7.16 and 7.17, by recalling the representation formula provided by Theorem 6.5 for the solution to (6.10). \square

Proof of 2. Recall that u is represented by

$$u(x) = \int_\Omega G_\Omega^s(x, y) \mu(y) dy + \int_{\partial\Omega} P_\Omega^s(x, y) \zeta(y) d\mathcal{H}(y).$$

By Point 1. and Lemma 7.11, $u \in C_{loc}^{2s+\alpha}(\Omega)$. Moreover, $u \in L^1(\Omega, \delta(x)dx)$ thanks to (6.13). So, we can compute pointwise $(-\Delta|_\Omega)^s u$ by using (6.3) and (8.6): this entails by the self-adjointness of the operator in (8.6) that

$$\int_\Omega (-\Delta|_\Omega)^s u \psi = \int_\Omega u (-\Delta|_\Omega)^s \psi = \int_\Omega \mu \psi, \quad \text{for any } \psi \in C_c^\infty(\Omega)$$

and we must conclude that $(-\Delta|_\Omega)^s u = \mu$ a.e. By continuity the equality holds everywhere.

We turn now to the boundary trace. The contribution given by $\mathbb{G}_\Omega^s \mu$ is irrelevant, because it is a bounded function as it follows from

$$|\mathbb{G}_\Omega^s \mu(x)| \leq C \|\mu\|_{L^\infty(\Omega)} \int_\Omega \frac{dy}{|x-y|^{N-2s}}$$

where we have used (7.12). Therefore, by Proposition 7.15, there also holds for all $x_0 \in \partial\Omega$,

$$\lim_{x \rightarrow x_0, x \in \Omega} \frac{u(x)}{h_1(x)} = \lim_{x \rightarrow x_0, x \in \Omega} \frac{\mathbb{G}_\Omega^s \mu(x) + \mathbb{P}_\Omega^s \zeta(x)}{h_1(x)} = \zeta(x_0).$$

\square

Proof of 3. By Lemma 7.12 $\mu \in C_{loc}^\varepsilon(\Omega)$. In addition, we have assumed that $\zeta \in C(\partial\Omega)$. Consider

$$v(x) = \int_\Omega G_\Omega^s(x, y) \mu(y) dy + \int_{\partial\Omega} P_\Omega^s(x, z) \zeta(z) d\mathcal{H}(z)$$

the weak solution associated to data μ and ζ . By the previous point of the Lemma, v is a classical solution to the equation, so that in a pointwise sense it holds

$$(-\Delta|_\Omega)^s(u - v) = 0 \text{ in } \Omega, \quad \frac{u - v}{h_1} = 0 \text{ on } \partial\Omega.$$

By applying Lemma 7.19 we conclude that $|u - v| \leq \varepsilon h_1$ for any $\varepsilon > 0$ and thus $u - v \equiv 0$. \square

Nonlinear problems and large solutions for the power nonlinearity

8.1 THE NONLINEAR PROBLEM

LEMMA 8.1: KATO'S INEQUALITY | For $f \in L^1(\Omega, \delta(x)dx)$ let $w \in L^1(\Omega, \delta(x)dx)$ weakly solve

$$\begin{cases} (-\Delta|_{\Omega})^s w = f & \text{in } \Omega \\ \frac{w}{h_1} = 0 & \text{on } \partial\Omega. \end{cases}$$

For any convex $\Phi : \mathbb{R} \rightarrow \mathbb{R}$, $\Phi \in C^2(\mathbb{R})$ such that $\Phi(0) = 0$ and $\Phi(w) \in L^1_{loc}(\Omega)$, it holds

$$(-\Delta|_{\Omega})^s \Phi(w) \leq \Phi'(w)(-\Delta|_{\Omega})^s w.$$

Moreover, the same holds for $\Phi(t) = t^+ = t \wedge 0$.

Proof. Let us first assume that $f \in C^\alpha_{loc}(\Omega)$. In this case, by Lemma 7.11, $w \in C^{2s+\alpha}_{loc}(\Omega)$ and the equality $(-\Delta|_{\Omega})^s w = f$ holds in a pointwise sense. Then

$$\begin{aligned} (-\Delta|_{\Omega})^s \Phi \circ w(x) &= \int_{\Omega} [\Phi(w(x)) - \Phi(w(y))] J(x, y) dy + \kappa(x) \Phi(w(x)) \\ &= \Phi'(w(x)) \int_{\Omega} [w(x) - w(y)] J(x, y) dy + \kappa(x) \Phi(w(x)) \\ &\quad - \int_{\Omega} [w(x) - w(y)]^2 J(x, y) \int_0^1 \Phi''(w(x) + t[w(y) - w(x)])(1-t) dt dy \\ &\leq \Phi'(w(x)) (-\Delta|_{\Omega})^s w(x) \end{aligned}$$

where we have used that $\Phi'' \geq 0$ in \mathbb{R} and that $\Phi'(t) \leq t\Phi'(t)$, which follows from $\Phi(0) = 0$.

We deal now with $f \in L^\infty(\Omega)$. Pick $\{f_j\}_{j \in \mathbb{N}} \subseteq C^\infty_c(\Omega)$ converging to f in $L^1(\Omega, \delta(x), dx)$ and bounded in $L^\infty(\Omega)$. The corresponding $\{w_j = \mathbb{G}^s_{\Omega} f_j\}_{j \in \mathbb{N}}$ converges to w in $L^1(\Omega, \delta(x)dx)$, is bounded in $L^\infty(\Omega)$ and without loss of generality we assume that $f_j \rightarrow f$ and $w_j \rightarrow w$ a.e. in Ω . We know that for any $\psi \in \mathcal{T}(\Omega)$, $\psi \geq 0$

$$\int_{\Omega} \Phi(w_j) (-\Delta|_{\Omega})^s \psi \leq \int_{\Omega} f_j \Phi'(w_j) \psi.$$

By the continuity of Φ and Φ' we have $\Phi(w_j) \rightarrow \Phi(w)$, $\Phi'(w_j) \rightarrow \Phi'(w)$ a.e. in Ω and that $\{\Phi(w_j)\}_{j \in \mathbb{N}}$, $\{\Phi'(w_j)\}_{j \in \mathbb{N}}$ are bounded in $L^\infty(\Omega)$. Since $\{f_j\}_{j \in \mathbb{N}}$ is converging to f in $L^1(\Omega, \delta(x)dx)$, then

$$\int_{\Omega} \Phi(w_j) (-\Delta|_{\Omega})^s \psi \longrightarrow \int_{\Omega} \Phi(w) (-\Delta|_{\Omega})^s \psi \quad \text{and} \quad \int_{\Omega} f_j \Phi'(w_j) \psi \longrightarrow \int_{\Omega} f \Phi'(w) \psi$$

by dominated convergence.

For a general $f \in L^1(\Omega, \delta(x)dx)$ define $f_{j,k} := (f \wedge j) \vee (-k)$, $j, k \in \mathbb{N}$. Also, split the expression of $\Phi = \Phi_1 - \Phi_2$ into the difference of two increasing function: this can be done in the following way. The function Φ' is continuous and increasing in \mathbb{R} , so that it can either have constant sign or there exists $t_0 \in \mathbb{R}$ such that $\Phi'(t_0) = 0$. If it has constant sign than Φ can be increasing or decreasing and we can choose respectively $\Phi_1 = \Phi, \Phi_2 = 0$ or $\Phi_1 = 0, \Phi_2 = -\Phi$. Otherwise we can take

$$\Phi_1(t) = \begin{cases} \Phi(t) & t > t_0 \\ \Phi(t_0) & t \leq t_0 \end{cases} \quad \text{and} \quad \Phi_2(t) = \begin{cases} 0 & t > t_0 \\ \Phi(t_0) - \Phi(t) & t \leq t_0 \end{cases}.$$

We already know that for any $\psi \in \mathcal{T}(\Omega)$, $\psi \geq 0$

$$\int_{\Omega} \Phi(w_{j,k}) (-\Delta|_{\Omega})^s \psi \leq \int_{\Omega} f_{j,k} \Phi'(w_{j,k}) \psi.$$

On the right-hand side we can use twice the monotone convergence, letting $j \uparrow \infty$ first and then $k \uparrow \infty$. On the left hand side, by writing $\Phi = \Phi_1 - \Phi_2$ again we can exploit several times the monotone convergence by splitting

$$\begin{aligned} \int_{\Omega} \Phi(w_{j,k}) (-\Delta|_{\Omega})^s \psi &= \int_{\Omega} \Phi_1(w_{j,k}) [(-\Delta|_{\Omega})^s \psi]^+ - \int_{\Omega} \Phi_1(w_{j,k}) [(-\Delta|_{\Omega})^s \psi]^- + \\ &\quad - \int_{\Omega} \Phi_2(w_{j,k}) [(-\Delta|_{\Omega})^s \psi]^+ + \int_{\Omega} \Phi_2(w_{j,k}) [(-\Delta|_{\Omega})^s \psi]^- \end{aligned}$$

to deduce the thesis.

Finally, note that $\Phi(t) = t^+$ can be monotonically approximated by

$$\Phi_j(t) = \frac{1}{2} \sqrt{t^2 + \frac{1}{j^2}} + \frac{t}{2} - \frac{1}{2j}.$$

which is convex, C^2 and $\Phi_j(0) = 0$. So

$$\int_{\Omega} \Phi_j(w) (-\Delta|_{\Omega})^s \psi \leq \int_{\Omega} f \Phi'_j(w) \psi.$$

Since $\Phi_j(t) \uparrow t^+$ and $2\Phi'_j(t) \uparrow 1 + \text{sgn}(t) = 2\chi_{(0,+\infty)}(t)$, we prove the last statement of the Lemma. \square

THEOREM 8.2 | *Let $f(x, t) : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function. Assume that there exists a subsolution and a supersolution $\underline{u}, \bar{u} \in L^1(\Omega, \delta(x)dx) \cap L_{loc}^{\infty}(\Omega)$ to*

$$\begin{cases} (-\Delta|_{\Omega})^s u = f(x, u) & \text{in } \Omega, \\ \frac{u}{h_1} = 0 & \text{on } \partial\Omega. \end{cases} \quad (8.1)$$

Assume in addition that $f(\cdot, v) \in L^1(\Omega, \delta(x)dx)$ for every $v \in L^1(\Omega, \delta(x)dx)$ such that $\underline{u} \leq v \leq \bar{u}$ a.e. Then, there exist weak solutions $u_1, u_2 \in L^1(\Omega, \delta(x)dx)$ in $[\underline{u}, \bar{u}]$ such that any solution u in the interval $[\underline{u}, \bar{u}]$ satisfies

$$\underline{u} \leq u_1 \leq u \leq u_2 \leq \bar{u} \quad \text{a.e.}$$

Moreover, if the nonlinearity f is decreasing in the second variable, then the solution is unique.

Proof. According to Montenegro and Ponce [67], the mapping $v \mapsto F(\cdot, v)$, where

$$F(x, t) := f(x, [t \wedge \bar{u}(x)] \vee \underline{u}(x)), \quad x \in \Omega, t \in \mathbb{R},$$

acts continuously from $L^1(\Omega, \delta(x)dx)$ into itself. In addition, the operator

$$\begin{aligned} \mathcal{K} : L^1(\Omega, \delta(x)dx) &\longrightarrow L^1(\Omega, \delta(x)dx) \\ v(x) &\longmapsto \mathcal{K}(v)(x) = \int_{\Omega} G_{|\Omega|}^s(x, y) F(y, v(y)) dy \end{aligned}$$

is compact. Indeed, take a bounded sequence $\{v_n\}_{n \in \mathbb{N}}$ in $L^1(\Omega, \delta(x)dx)$. On a compact set $K \subset \subset \Omega$, \underline{u}, \bar{u} are essentially bounded and so must be the sequence $\{F(\cdot, v_n)\}_{n \in \mathbb{N}}$. By Theorem 6.5 and Lemma 7.11, $\{\mathcal{K}(v_n)\}_{n \in \mathbb{N}}$ is bounded in $C_{loc}^\alpha(K) \cap L^p(\Omega, \delta(x)dx)$, $p \in [1, (N+1)/(N+1-2s))$. In particular, a subsequence $\{v_{n_k}\}_{k \in \mathbb{N}}$ converges locally uniformly to some v . By Hölder's inequality, we also have

$$\|v_{n_k} - v\|_{L^1(\Omega \setminus K, \delta(x)dx)} \leq \|v_{n_k} - v\|_{L^p(\Omega \setminus K, \delta(x)dx)} \|\chi_{\Omega \setminus K}\|_{L^{p'}(\Omega \setminus K, \delta(x)dx)}.$$

Hence,

$$\|v_{n_k} - v\|_{L^1(\Omega, \delta(x)dx)} \leq \|v_{n_k} - v\|_{L^\infty(K, \delta(x)dx)} \|\delta\|_{L^1(\Omega)} + C \|\chi_{\Omega \setminus K}\|_{L^{p'}(\Omega \setminus K, \delta(x)dx)}.$$

Letting $k \rightarrow +\infty$ and then $K \rightarrow \Omega$, we deduce that \mathcal{K} is compact and by the Schauder's Fixed Point Theorem, \mathcal{K} has a fixed point $u \in L^1(\Omega, \delta(x)dx)$. We then may prove that $\underline{u} \leq u \leq \bar{u}$ by means of the Kato's Inequality (Lemma 8.1) as it is done in [67], which yields that u is a solution of (8.1).

The proof of the existence of the minimal and a maximal solution $u_1, u_2 \in L^1(\Omega, \delta(x)dx)$ can be performed in an analogous way as in [67], as the only needed tool is the Kato's Inequality.

As for the uniqueness, suppose f is decreasing in the second variable and consider two solutions $u, v \in L^1(\Omega, \delta(x)dx)$ to (8.1). By the Kato's Inequality Lemma 8.1, we have

$$(-\Delta|_\Omega)^s(u - v)^+ \leq \chi_{\{u > v\}}[f(x, u) - f(x, v)] \leq 0 \quad \text{in } \Omega$$

which implies $(u - v)^+ \leq 0$ by the Maximum Principle Lemma 7.20. Reversing the roles of u and v , we get also $(v - u)^+ \leq 0$, thus $u \equiv v$ in Ω . \square

8.1.1 Proof of Theorem 6.6

Problem (6.17) is equivalent to

$$\begin{cases} (-\Delta|_\Omega)^s v = g(x, \mathbb{P}_\Omega^s \zeta - v) & \text{in } \Omega \\ \frac{v}{h_1} = 0 & \text{on } \partial\Omega \end{cases} \quad (8.2)$$

that possesses $\bar{u} = \mathbb{P}_\Omega^s \zeta$ as a supersolution and $\underline{u} = 0$ as a subsolution. Indeed, by equation (7.20) we have

$$0 \leq \mathbb{P}_\Omega^s \zeta \leq \|\zeta\|_{L^\infty(\Omega)} h_1 \leq C \|\zeta\|_{L^\infty(\Omega)} \delta^{-(2-2s)}.$$

Thus any $v \in L^1(\Omega, \delta(x)dx)$ such that $0 \leq v \leq \mathbb{P}_\Omega^s \zeta$ satisfies

$$g(x, v) \leq h(v) \leq h(c\delta^{-(2-2s)}) \in L^1(\Omega, \delta(x)dx).$$

So, all hypotheses of Theorem 8.2 are satisfied and the result follows. \square

8.2 LARGE SOLUTIONS

Consider the sequence $\{u_j\}_{j \in \mathbb{N}}$ built by solving

$$\begin{cases} (-\Delta|_\Omega)^s u_j = -u_j^p & \text{in } \Omega \\ \frac{u_j}{h_1} = j & \text{on } \partial\Omega. \end{cases} \quad (8.3)$$

Theorem 6.6 guarantees the existence of such a sequence if $\delta^{-(2-2s)p} \in L^1(\Omega, \delta(x)dx)$, i.e. $p < 1/(1-s)$. We claim that the sequence is increasing in Ω : indeed the solution to problem (8.3) is a subsolution for the same problem with boundary datum $j+1$. In view of this, the sequence $\{u_j\}_{j \in \mathbb{N}}$ admits a pointwise limit, possibly infinite.

8.2.1 Construction of a supersolution

LEMMA 8.3 | *There exist $\delta_0, C > 0$ such that*

$$(-\Delta|_{\Omega})^s \delta^{-\alpha} \geq -C \delta^{-\alpha p}, \quad \text{for } \delta < \delta_0 \text{ and } \alpha = \frac{2s}{p-1}.$$

Proof. We use the expression in equation (6.3). Obviously,

$$(-\Delta|_{\Omega})^s \delta^{-\alpha}(x) = \int_{\Omega} [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x, y) dy + \delta(x)^{-\alpha} \kappa(x) \geq \int_{\Omega} [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x, y) dy.$$

For any fixed $x \in \Omega$ close to $\partial\Omega$, split the domain Ω into three parts:

$$\begin{aligned} \Omega_1 &= \left\{ y \in \Omega : \delta(y) \geq \frac{3}{2} \delta(x) \right\}, \\ \Omega_2 &= \left\{ y \in \Omega : \frac{1}{2} \delta(x) < \delta(y) < \frac{3}{2} \delta(x) \right\}, \\ \Omega_3 &= \left\{ y \in \Omega : \delta(y) \leq \frac{1}{2} \delta(x) \right\}. \end{aligned}$$

For $y \in \Omega_1$, since $\delta(y) > \delta(x)$, it holds $\delta(x)^{-\alpha} - \delta(y)^{-\alpha} > 0$ and we can drop the integral on Ω_1 . Also, since it holds by equation (7.13)

$$J(x, y) \leq \frac{C}{|x - y|^{N+2s}},$$

the integration on Ω_2 can be performed as in *Second step* in Proposition 4.2 providing

$$PV \int_{\Omega_2} [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x, y) dy \geq -C \delta(x)^{-\alpha-2s} = -C \delta(x)^{-\alpha p}.$$

To integrate on Ω_3 we exploit once again (7.13) under the form

$$J(x, y) \leq C \cdot \frac{\delta(x) \delta(y)}{|x - y|^{N+2+2s}}$$

to deduce

$$\int_{\Omega_3} [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x, y) dy \geq -C \delta(x) \int_{\Omega_3} \frac{\delta(y)^{1-\alpha}}{|x - y|^{N+2+2s}}.$$

Again, a direct computation as in *Third step* in Proposition 4.2 yields

$$\int_{\Omega_3} [\delta(x)^{-\alpha} - \delta(y)^{-\alpha}] J(x, y) dy \geq -C \delta(x)^{-\alpha-2s} = -C \delta(x)^{-\alpha p}.$$

□

LEMMA 8.4 | *If a function $v \in L^1(\Omega, \delta(x)dx)$ satisfies*

$$(-\Delta|_{\Omega})^s v \in L_{loc}^{\infty}(\Omega), \quad (-\Delta|_{\Omega})^s v(x) \geq -C v(x)^p, \quad \text{when } \delta(x) < \delta_0, \quad (8.4)$$

for some $C, \delta_0 > 0$, then there exists $\bar{u} \in L^1(\Omega, \delta(x)dx)$ such that

$$(-\Delta|_{\Omega})^s \bar{u}(x) \geq -\bar{u}(x)^p, \quad \text{throughout } \Omega. \quad (8.5)$$

Proof. Let $\lambda := C^{1/(p-1)} \vee 1$ and $\Omega_0 = \{x \in \Omega : \delta(x) < \delta_0\}$, then

$$(-\Delta|_{\Omega})^s (\lambda v) \geq -(\lambda v)^p, \quad \text{in } \Omega_0.$$

Let also $\mu := \lambda \|(-\Delta|_{\Omega})^s v\|_{L^{\infty}(\Omega \setminus \Omega_0)}$ and define $\bar{u} = \mu \mathbb{G}_{\Omega}^s 1 + \lambda v$. On \bar{u} we have

$$(-\Delta|_{\Omega})^s \bar{u} = \mu + \lambda (-\Delta|_{\Omega})^s v \geq \lambda |(-\Delta|_{\Omega})^s v| + \lambda (-\Delta|_{\Omega})^s v \geq -\bar{u}^p \quad \text{throughout } \Omega.$$

□

COROLLARY 8.5 | *There exists a function $\bar{u} \in L^1(\Omega, \delta(x)dx)$ such that the inequality*

$$(-\Delta|_{\Omega})^s \bar{u} \geq -\bar{u}^p, \quad \text{in } \Omega,$$

holds in a pointwise sense. Moreover, $\bar{u} \asymp \delta^{-2s/(p-1)}$.

Proof. Apply Lemma 8.4 with $v = \delta^{-2s/(p-1)}$: the corresponding \bar{u} will be of the form

$$\bar{u} = \mu \mathbb{G}_{|\Omega}^s 1 + \lambda \delta^{-2s/(p-1)}.$$

□

8.2.2 Existence

LEMMA 8.6 | *For any $j \in \mathbb{N}$, the solution u_j to problem (8.3) satisfies the upper bound*

$$u_j \leq \bar{u}, \quad \text{in } \Omega,$$

where \bar{u} is provided by Corollary 8.5.

Proof. Write $u_j = jh_1 - v_j$ where

$$\begin{cases} (-\Delta|_{\Omega})^s v_j = (jh_1 - v_j)^p & \text{in } \Omega \\ \frac{v_j}{h_1} = 0 & \text{on } \partial\Omega. \end{cases}$$

and $0 \leq v_j \leq jh_1$. Since $(jh_1 - v_j)^p \in L_{loc}^{\infty}(\Omega)$, we deduce that $v_j \in C_{loc}^{\alpha}(\Omega)$ for any $\alpha \in (0, 2s)$. By bootstrapping $v_j \in C^{\infty}(\Omega)$ and, by Lemma 7.11, also $u_j \in C^{\infty}(\Omega)$. This says that u_j is a classical solution to problem (8.3). Now, we have that, by the boundary behaviour of \bar{u} stated in Corollary 8.5, $u_j \leq \bar{u}$ close enough to $\partial\Omega$ (depending on the value of j) and

$$(-\Delta|_{\Omega})^s (\bar{u} - u_j) \geq u_j^p - \bar{u}^p, \quad \text{in } \Omega.$$

Since $u_j^p - \bar{u}^p \in C(\Omega)$ and $\lim_{x \rightarrow \partial\Omega} u_j^p - \bar{u}^p = -\infty$, there exists $x_0 \in \Omega$ such that $u_j(x_0)^p - \bar{u}(x_0)^p = m =: \max_{x \in \Omega} (u_j(x)^p - \bar{u}(x)^p)$. If $m > 0$ then $(-\Delta|_{\Omega})^s (\bar{u} - u_j)(x_0) \geq m > 0$: this is a contradiction, as Definition 6.3 implies. Thus $m \leq 0$ and $u_j \leq \bar{u}$ throughout Ω . □

THEOREM 8.7 | *For any $p \in \left(1 + s, \frac{1}{1-s}\right)$ there exists a function $u \in L^1(\Omega, \delta(x)dx)$ solving*

$$\begin{cases} (-\Delta|_{\Omega})^s u = -u^p & \text{in } \Omega \\ \delta^{2-2s} u = +\infty & \text{on } \partial\Omega \end{cases}$$

both in a distributional and pointwise sense.

Proof. Consider the sequence $\{u_j\}_{j \in \mathbb{N}}$ provided by problem (8.3): it is increasing and locally bounded by Lemma 8.6, so it has a pointwise limit $u \leq \bar{u}$, where \bar{u} is the function provided by Corollary 8.5. Since $p > 1 + s$ and $\bar{u} \leq C\delta^{-2s/(p-1)}$, then $u \in L^1(\Omega, \delta(x)dx)$. Pick now $\psi \in C_c^{\infty}(\Omega)$, and recall that $\delta^{-1}(-\Delta|_{\Omega})^s \psi \in L^{\infty}(\Omega)$: we have, by dominated convergence,

$$\int_{\Omega} u_j (-\Delta|_{\Omega})^s \psi \xrightarrow{j \uparrow \infty} \int_{\Omega} u (-\Delta|_{\Omega})^s \psi, \quad \int_{\Omega} u_j^p \psi \xrightarrow{j \uparrow \infty} \int_{\Omega} u^p \psi$$

so we deduce

$$\int_{\Omega} u (-\Delta|_{\Omega})^s \psi = - \int_{\Omega} u^p \psi.$$

Note now that for any compact $K \subset \subset \Omega$, applying Lemma 7.11 we get for any $\alpha \in (0, 2s)$

$$\|u_j\|_{C^{\alpha}(K)} \leq C \left(\|u_j\|_{L^{\infty}(K)}^p + \|u_j\|_{L^1(\Omega, \delta(x)dx)} \right) \leq C \left(\|\bar{u}\|_{L^{\infty}(K)}^p + \|\bar{u}\|_{L^1(\Omega, \delta(x)dx)} \right)$$

which means that $\{u_j\}_{j \in \mathbb{N}}$ is equibounded and equicontinuous in $C(K)$. By the Ascoli-Arzelà Theorem, its pointwise limit u will be in $C(K)$ too. Now, since

$$(-\Delta|_{\Omega})^s u = -u^p \quad \text{in } \mathcal{D}'(\Omega),$$

by bootstrapping the interior regularity in Lemma 7.11, we deduce $u \in C^\infty(\Omega)$. So, its spectral fractional Laplacian is pointwise well-defined and the equation is satisfied in a pointwise sense. Also,

$$\liminf_{x \rightarrow \partial\Omega} \frac{u(x)}{h_1(x)} \geq \liminf_{x \rightarrow \partial\Omega} \frac{u_j(x)}{h_1(x)} = j$$

and we obtain the desired boundary datum. \square

8.3 APPENDIX

8.3.1 Another representation for the spectral fractional Laplacian

LEMMA 8.8 | For any $u \in H(2s)$ and almost every $x \in \Omega$, there holds

$$(-\Delta|_{\Omega})^s u(x) = PV \int_{\Omega} [u(x) - u(y)] J(x, y) dy + \kappa(x) u(x),$$

where $J(x, y)$ and $\kappa(x)$ are given by (6.4).

Proof. Assume that $u = \varphi_j$ is an eigenfunction of the Dirichlet Laplacian associated to the eigenvalue λ_j . Then, $(-\Delta|_{\Omega})^s u = \lambda_j^s u$, $e^{t\Delta|_{\Omega}} u = \int_{\Omega} p_{\Omega}(t, \cdot, y) u(y) dy = e^{-t\lambda_j} u$ and for all $x \in \Omega$

$$\begin{aligned} \frac{\Gamma(1-s)}{s} (-\Delta|_{\Omega})^s u(x) &= \\ &= \int_0^\infty \left(u(x) - e^{t\Delta|_{\Omega}} u(x) \right) \frac{dt}{t^{1+s}} \end{aligned} \quad (8.6)$$

$$\begin{aligned} &= \int_0^\infty \left(u(x) - \int_{\Omega} p_{\Omega}(t, x, y) u(y) dy \right) \frac{dt}{t^{1+s}} \\ &= \lim_{\epsilon \rightarrow 0} \int_0^\infty \int_{\Omega \setminus B(x, \epsilon)} p_{\Omega}(t, x, y) [u(x) - u(y)] dy \frac{dt}{t^{1+s}} + \int_0^\infty u(x) \left(1 - \int_{\Omega} p_{\Omega}(t, x, y) dy \right) \frac{dt}{t^{1+s}} \\ &= PV \int_{\Omega} [u(x) - u(y)] J(x, y) dy + \kappa(x) u(x). \end{aligned} \quad (8.7)$$

By linearity, equality holds on the linear span of the eigenvectors. Now, if $u \in H(2s)$, a sequence $\{u_n\}_{n \in \mathbb{N}}$ of functions belonging to that span converges to u in $H(2s)$. In particular, $(-\Delta|_{\Omega})^s u_n$ converges to $(-\Delta|_{\Omega})^s u$ in $L^2(\Omega)$. Note also that for $v \in L^2(\Omega)$,

$$\frac{s}{\Gamma(1-s)} \left| \int_{\Omega} \int_0^\infty \left(u(x) - e^{t\Delta|_{\Omega}} u(x) \right) \frac{dt}{t^{1+s}} \cdot v(x) dx \right| = \left| \sum_{k=1}^{+\infty} \lambda_k^s \widehat{u}_k \widehat{v}_k \right| \leq \|u\|_{H(2s)} \|v\|_{L^2(\Omega)}$$

so that we may also pass to the limit in $L^2(\Omega)$ when computing (8.6) along the sequence $\{u_n\}_{n \in \mathbb{N}}$. By the Fubini's theorem, for almost every $x \in \Omega$, all subsequent integrals are convergent and the identities remain valid. \square

8.3.2 The reduction to the flat case

In this paragraph we are going to justify the computation of the asymptotic behaviour of integrals of the type

$$\int_{A \cap \Omega} F(\delta(x), \delta(y), |x - y|) dy \quad \text{as } \delta(x) \downarrow 0,$$

where A is a fixed neighbourhood of x with $A \cap \partial\Omega \neq \emptyset$, by just looking at

$$\int_0^1 dt \int_B dy' F\left(\delta(x), t, \sqrt{|y'|^2 + |t - \delta(x)|^2}\right).$$

The first thing to be proved is that

$$|x - y|^2 \asymp |x_0 - y_0|^2 + |\delta(x) - \delta(y)|^2,$$

where x_0, y_0 are respectively the projections of x, y on $\partial\Omega$.

LEMMA 8.9 | *There exists $\varepsilon = \varepsilon(\Omega) > 0$ such that for any $x \in \Omega$, $x = x_0 + \delta(x)\nabla\delta(x_0)$, $x_0 \in \partial\Omega$, with $\delta(x) < \varepsilon$ and any $y \in \Omega$ with $\delta(y) < \varepsilon$ and $|y_0 - x_0| < \varepsilon$*

$$\frac{1}{2} (|x_0 - y_0|^2 + |\delta(x) - \delta(y)|^2) \leq |x - y|^2 \leq \frac{3}{2} (|x_0 - y_0|^2 + |\delta(x) - \delta(y)|^2).$$

Proof. Call $\Omega_\varepsilon = \{x \in \Omega : \delta(x) < \varepsilon\}$. Write $x = x_0 + \delta(x)\nabla\delta(x_0)$, $y = y_0 + \delta(y)\nabla\delta(y_0)$. Then

$$\begin{aligned} |x - y|^2 &= |x_0 - y_0|^2 + |\delta(x) - \delta(y)|^2 + \delta(y)^2 |\nabla\delta(x_0) - \nabla\delta(y_0)|^2 + 2[\delta(x) - \delta(y)] \langle x_0 - y_0, \nabla\delta(x_0) \rangle + \\ &\quad + 2\delta(y) \langle x_0 - y_0, \nabla\delta(x_0) - \nabla\delta(y_0) \rangle + 2\delta(y)[\delta(x) - \delta(y)] \langle \nabla\delta(x_0), \nabla\delta(x_0) - \nabla\delta(y_0) \rangle. \end{aligned} \quad (8.8)$$

Since, for $\varepsilon > 0$ small, $\delta \in C^{1,1}(\Omega_\varepsilon)$ and

$$\begin{aligned} |\nabla\delta(x) - \nabla\delta(y)|^2 &\leq \|\delta\|_{C^{1,1}(\Omega_\varepsilon)}^2 |x - y|^2 \\ \langle x_0 - y_0, \nabla\delta(x_0) \rangle &= O(|x_0 - y_0|^2) \\ |\langle x_0 - y_0, \nabla\delta(x_0) - \nabla\delta(y_0) \rangle| &\leq \|\delta\|_{C^{1,1}(\Omega_\varepsilon)} |x_0 - y_0|^2 \\ |\delta(x) - \delta(y)| &= \|\delta\|_{C^{1,1}(\Omega_\varepsilon)} |x - y| \\ |\langle \nabla\delta(x_0), \nabla\delta(x_0) - \nabla\delta(y_0) \rangle| &\leq \|\delta\|_{C^{1,1}(\Omega_\varepsilon)} |x_0 - y_0| \end{aligned}$$

The error term we obtained in (8.8) can be reabsorbed in the other ones by choosing $\varepsilon > 0$ small enough to have

$$\begin{aligned} \delta(y)^2 |\nabla\delta(x_0) - \nabla\delta(y_0)|^2 + 2|\delta(x) - \delta(y)| \cdot |\langle x_0 - y_0, \nabla\delta(x_0) \rangle| + 2\delta(y) |\langle x_0 - y_0, \nabla\delta(x_0) - \nabla\delta(y_0) \rangle| + \\ + 2\delta(y) |\delta(x) - \delta(y)| \cdot |\langle \nabla\delta(x_0), \nabla\delta(x_0) - \nabla\delta(y_0) \rangle| \leq \frac{1}{2} (|x_0 - y_0|^2 + |\delta(x) - \delta(y)|^2). \end{aligned}$$

□

LEMMA 8.10 | *Let $F : (0, +\infty)^3 \rightarrow (0, +\infty)$ be a continuous function, decreasing in the third variable and $\Omega_\varepsilon = \{x \in \Omega : \delta(x) < \varepsilon\}$, with $\varepsilon = \varepsilon(\Omega) > 0$ provided by the previous lemma. Consider $x = x_0 + \delta(x)\nabla\delta(x_0)$, $x_0 \in \partial\Omega$, and the neighbourhood A of the point x , defined by $A = \{y \in \Omega_\varepsilon : y = y_0 + \delta(y)\nabla\delta(y_0), |x_0 - y_0| < \varepsilon\}$. Then there exist constants $0 < c_1 < c_2$, $c_1 = c_1(\Omega)$, $c_2 = c_2(\Omega)$ such that*

$$\begin{aligned} c_1 \int_0^\varepsilon dt \int_{B'_\varepsilon} dy' F\left(\delta(x), t, c_2 \sqrt{|y'|^2 + |t - \delta(x)|^2}\right) &\leq \int_A F(\delta(x), \delta(y), |x - y|) dy \leq \\ &\leq c_2 \int_0^\varepsilon dt \int_{B'_\varepsilon} dy' F\left(\delta(x), t, c_1 \sqrt{|y'|^2 + |t - \delta(x)|^2}\right) \end{aligned}$$

where the ' superscript denotes objects that live in \mathbb{R}^{N-1} .

Proof. By writing $y = y_0 + \delta(y)\nabla\delta(y_0)$, $y_0 \in \partial\Omega$ and using the Fubini's Theorem, we can split the integration into the variables y_0 and $t = \delta(y)$:

$$\int_A F(\delta(x), \delta(y), |x - y|) dy \leq \int_{B_\varepsilon(x_0) \cap \partial\Omega} \left(\int_0^\varepsilon F(\delta(x), t, |x - y_0 - t\nabla\delta(y_0)|) dt \right) d\mathcal{H}(y_0).$$

Using the monotony of F and the above lemma, we get

$$\int_A F(\delta(x), \delta(y), |x - y|) dy \leq \int_{B_\varepsilon(x_0) \cap \partial\Omega} \left(\int_0^\varepsilon F\left(\delta(x), t, c\sqrt{|x_0 - y_0|^2 + |\delta(x) - t|^2}\right) dt \right) d\mathcal{H}(y_0)$$

where c is a universal constant. Representing $B_\varepsilon(x_0) \cap \partial\Omega$ via a diffeomorphism γ with a ball $B'_\varepsilon \subset \mathbb{R}^{N-1}$ centered at 0, we can transform the integration in the y_0 variable into the integration onto B'_ε . The volume element $|D\gamma|$ will be bounded above and below by

$$0 < c_1 \leq |D\gamma| \leq c_2,$$

in view of the smoothness assumptions on $\partial\Omega$. □

— Part III —

Nonlocal curvatures

Nonlocal directional curvatures

In this Chapter we study a nonlocal notion of curvature of a surface. For some bibliography dealing with the motion by mean curvature and minimal surfaces, see e.g. [4, 8, 24, 17, 19, 22, 21, 38, 45, 53, 75] as well as [85] for a recent review.

The Chapter is organized as follows: the introductory Section 9.1 recalls some basic definitions and facts on classical curvatures of smooth surfaces (of course, this part can be skipped by the expert reader but we included it in order to make a clear comparison between the classical setting and the nonlocal one); in Section 9.2 we introduce a definition of *nonlocal directional curvature* and give some ideas of the context in which it arises; finally, we state some theorems which compare similarities and differences between the local and the nonlocal setting. The remaining sections are devoted to proofs and explicit computations.

Though the motivation of the present Chapter arises in the framework of nonlocal minimal surfaces and integro-differential operators of fractional type, which are subjects that involve a very advanced technology, the Chapter itself is completely self-contained and no prior knowledge on the topic is required to follow the involved proofs. Also, we put an effort in keeping all the arguments as elementary as possible and accessible to a wide audience.

Notation

In the following we will always use:

- N to denote the dimension of the Euclidean space \mathbb{R}^N , with $N \geq 3$, whose points are sometimes written in the form $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$,
- CE to denote the complementary set of $E \subseteq \mathbb{R}^N$, i.e. $CE := \mathbb{R}^N \setminus E$,
- $\mathcal{H}^{N-2}(E)$ to denote the $(N-2)$ -dimensional Hausdorff measure of a set E ,
- S^{N-2} to denote the $(N-2)$ -dimensional unit sphere in \mathbb{R}^{N-1} , namely

$$S^{N-2} := \{e \in \mathbb{R}^{N-1} : |e| = 1\};$$

with a slight abuse of notation, we will also identify S^{N-2} and the set

$$\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| = 1, x_N = 0\} \subseteq \mathbb{R}^N :$$

note that the latter set is simply an $(N-2)$ -dimensional sphere lying in an $(N-1)$ -dimensional subspace of \mathbb{R}^N and this justifies our notation,

- ω_{N-2} to denote the $(N-2)$ -dimensional Hausdorff measure of the $(N-2)$ -dimensional sphere, that is $\omega_{N-2} := \mathcal{H}^{N-2}(S^{N-2})$,

- χ_E , where $E \subseteq \mathbb{R}^N$, for the characteristic function of E , i.e.

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \in \mathcal{C}E, \end{cases}$$

- $\tilde{\chi}_E$ for the difference $\chi_E - \chi_{\mathcal{C}E}$, namely

$$\tilde{\chi}_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ -1 & \text{if } x \in \mathcal{C}E, \end{cases}$$

- $\langle Ax, x \rangle$, when A is an $N \times N$ real symmetric matrix and $x \in \mathbb{R}^N$, to denote the quadratic form on \mathbb{R}^N represented by A and evaluated at x , i.e. if $A = \{A_{ij}\}_{i,j=1,\dots,N}$, then

$$\langle Ax, x \rangle := \sum_{i,j=1}^N A_{ij} x_i x_j,$$

- \int_{Ω} , when $\Omega \subseteq \mathbb{R}^N$ has finite measure, for the integral operator $\frac{1}{|\Omega|} \int_{\Omega}$.

Also, we will sometimes write multiple integrals by putting in evidence the integration variables according to the expression

$$\int_X dx \int_Y dy \int_Z dz f(x, y, z) \quad \text{in place of} \quad \int_X \left[\int_Y \left[\int_Z f(x, y, z) dz \right] dy \right] dx.$$

Moreover, we will reserve the name s for a fractional parameter that, in our scaling, is taken in $(0, 1/2)$.

9.1 SUMMARY ON CLASSICAL CURVATURES

In order to make a clear comparison between some classical facts and their corresponding nonlocal counterparts, we recall here a few basic results. Namely, some well-known facts on the classical concept of curvature show a nice and deep interplay between geometry, analysis and algebra that can risk to be not evident from the beginning. In particular, the mean curvature, which is a geometrical object, can be described in normal coordinates by the Laplacian, which comes from analysis, and also can be seen as the trace of a linear map, and here an algebraic notion shows up. The interplay between these disciplines has some striking consequences: let us recall two of them.

First of all, we recall that, given a C^2 surface Σ , a point $p \in \Sigma$ and a vector e in the tangent space of Σ at p , one may define the classical notion of *directional curvature of Σ at p in direction e* by the curvature at p of the curve C lying in the intersection between Σ and the two-dimensional plane spanned by e and the normal vector of Σ at p . We denote by K_e the directional curvature in direction e .

It is well-known that this directional curvature may be easily computed in normal coordinates. Namely, suppose we are given a set $E \subseteq \mathbb{R}^N$ such that $0 \in \partial E$, and suppose that $\Sigma = \partial E$ is described as a graph in normal coordinates, meaning that, in an open ball $B_r \subseteq \mathbb{R}^N$, ∂E coincides with the graph of a C^2 function $\varphi : B_r \cap \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ with $\varphi(0) = 0$ and $\nabla \varphi(0) = 0$. Then the directional curvature in direction e is given by

$$K_e = \langle D^2 \varphi(0) e, e \rangle = D_e^2 \varphi(0), \quad e \in \mathbb{R}^{N-1}, |e| = 1$$

where $D^2 \varphi(0)$ is the Hessian matrix of φ evaluated at 0.

Since $D^2 \varphi(0)$ is a real symmetric matrix, it will admit $N - 1$ real eigenvalues $\lambda_1 \leq \dots \leq \lambda_{N-1}$ called *principal curvatures*. Moreover, associated to these eigenvalues, there is an orthonormal basis of eigenvectors v_1, \dots, v_{N-1} called *principal directions*.

The arithmetic mean of the principal curvatures is called *mean curvature* and we denote it by H , namely

$$H := \frac{\lambda_1 + \dots + \lambda_{N-1}}{N-1}.$$

The above mentioned algebraic formulation implies that the principal directions v_1, \dots, v_{N-1} can be always chosen orthogonally, which is a somehow surprising geometric outcome that allows to easily compute any directional curvature once the principal curvatures are known:

THEOREM 9.1 | Let Σ be a C^2 surface described as the graph of the C^2 function $\varphi : B_r \cap \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ with $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. Every directional curvature can be calculated using principal curvatures, i.e. the eigenvalues $\lambda_1, \dots, \lambda_{N-1}$ of the matrix $D^2\varphi(0)$, and principal curvatures, i.e. an orthonormal basis v_1, \dots, v_{N-1} of eigenvectors; given a vector $e = \alpha_1 v_1 + \dots + \alpha_{N-1} v_{N-1}$, with $\alpha_1^2 + \dots + \alpha_{N-1}^2 = 1$, then

$$K_e = \langle D^2\varphi(0) e, e \rangle = \lambda_1 \alpha_1^2 + \dots + \lambda_{N-1} \alpha_{N-1}^2.$$

REMARK 9.2 | We point out that Theorem 9.1 implies also that all directional curvatures are bounded below by λ_1 and above by λ_{N-1} and that λ_1 and λ_{N-1} are attained along orthogonal directions. In particular, when $N = 3$, the two principal curvatures are the minimum and the maximum of the directional curvature K_e for $e \in S^1$.

REMARK 9.3 | In a sense, Theorem 9.1 shows a sort of linear phenomenon that drives the classical directional curvatures. As we will see in forthcoming Remark 9.9, this linear feature cannot be recovered in the case of nonlocal directional curvatures, that are somewhat intrinsically nonlinear in nature.

Furthermore, the spherical average of directional curvatures may be reconstructed by the arithmetic mean of the principal curvatures, that is the normalized integral of K_e over $e \in S^{N-2}$ coincides with the normalized trace of the Hessian matrix, thus reducing the (difficult, in general) computation of an integral on the sphere to a (simple, in general) sum of finitely many terms (that are the eigenvalues of the Hessian matrix) and this clearly provides an important computational simplification:

THEOREM 9.4 | Let Σ be a C^2 surface described as the graph of the C^2 function $\varphi : B_r \cap \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ with $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. The mean curvature can be defined

$$H := \frac{\lambda_1 + \dots + \lambda_{N-1}}{N-1} \quad \text{or equivalently} \quad H := \int_{S^{N-2}} K_e d\mathcal{H}^{N-2}(e),$$

since

$$\int_{S^{N-2}} K_e d\mathcal{H}^{N-2}(e) = \int_{S^{N-2}} \langle D^2\varphi(0) e, e \rangle d\mathcal{H}^{N-2}(e) = \frac{\lambda_1 + \dots + \lambda_{N-1}}{N-1}.$$

Proof. By symmetry

$$\int_{S^{N-2}} \alpha_i^2 d\mathcal{H}^{N-2}(\alpha) = \int_{S^{N-2}} \alpha_1^2 d\mathcal{H}^{N-2}(\alpha) \quad (9.1)$$

for any $i \in \{1, \dots, N-1\}$. By summing up in (9.1) we obtain

$$\omega_{N-2} = \int_{S^{N-2}} 1 d\mathcal{H}^{N-2}(\alpha) = \int_{S^{N-2}} \sum_{i=1}^{N-1} \alpha_i^2 d\mathcal{H}^{N-2}(\alpha) = (N-1) \int_{S^{N-2}} \alpha_1^2 d\mathcal{H}^{N-2}(\alpha).$$

Using again (9.1) we deduce from the identity above that

$$\omega_{N-2} = (N-1) \int_{S^{N-2}} \alpha_i^2 d\mathcal{H}^{N-2}(\alpha)$$

for any $i \in \{1, \dots, N-1\}$. As a consequence

$$\begin{aligned} \frac{1}{\omega_{N-2}} \int_{S^{N-2}} \langle D^2\varphi(0) e, e \rangle d\mathcal{H}^{N-2}(e) &= \frac{1}{\omega_{N-2}} \int_{S^{N-2}} (\lambda_1 \alpha_1^2 + \dots + \lambda_{N-1} \alpha_{N-1}^2) d\mathcal{H}^{N-2}(\alpha) = \\ &= \frac{1}{\omega_{N-2}} \sum_{i=1}^{N-1} \lambda_i \int_{S^{N-2}} \alpha_i^2 d\mathcal{H}^{N-2}(\alpha) = \frac{\lambda_1 + \dots + \lambda_{N-1}}{\omega_{N-2}} \cdot \frac{\omega_{N-2}}{N-1} \end{aligned}$$

as desired. \square

Associated with these concepts, there is also a theory of *motion by mean curvature*. Let us think of a bounded set $\Omega \subseteq \mathbb{R}^N$ whose shape changes in time according to local features of its boundary, i.e. each point x_0 of the boundary moves along the normal direction to $\partial\Omega$ at x_0 and with a speed given by the mean curvature of $\partial\Omega$ at x_0 . In [66], with the aid of [54] and [42], it is possible to find the following approximation of this motion. Let evolve the function $\tilde{\chi}_\Omega$ according to the heat equation

$$\begin{cases} \partial_t u(x, t) = \Delta u(x, t), \\ u(x, 0) = \tilde{\chi}_\Omega(x). \end{cases} \quad (9.2)$$

Then the set $\Omega_\varepsilon = \{u(x, \varepsilon) > 0\}$, for small $\varepsilon > 0$, has a boundary close to the evolution of $\partial\Omega$ by mean curvature. Before passing to the nonlocal case, we would like to bring to the reader's attention two facts:

1. we underline how the evolution of a point $x_0 \in \partial\Omega$ depends only on the shape of Ω in a neighborhood of x_0 ,
2. we recall that if a set E has minimal perimeter in a region U , then it has zero mean curvature at each point of $\partial E \cap U$, see [46], and we can say that this is the Euler-Lagrange equation associated to the minimization of the perimeter of a set; therefore a set with minimal perimeter will be a stationary solution to the motion by mean curvature.

9.2 NONLOCAL DIRECTIONAL CURVATURES

From now on we take $s \in (0, 1/2)$ and a set $E \subseteq \mathbb{R}^N$, with C^2 boundary ∂E .

9.2.1 Nonlocal definitions

We introduce here the nonlocal objects that will play the role of directional and mean curvatures (for details, heuristics and justifications of our definitions see Section 9.2.2).

DEFINITION 9.5: NONLOCAL MEAN CURVATURE | *The nonlocal mean curvature of ∂E at the point $p \in \partial E$ is*

$$H_s := \frac{1}{\omega_{N-2}} \int_{\mathbb{R}^N} \frac{\tilde{\chi}_E(x)}{|x-p|^{N+2s}} dx. \quad (9.3)$$

Denote now by ν be a normal unit vector for ∂E at p . Let also e be any unit vector in the tangent space of ∂E at p and¹ let $\pi(e)$ the two-dimensional open half-plane

$$\pi(e) := \{y \in \mathbb{R}^N : y = \rho e + h\nu, \rho > 0, h \in \mathbb{R}\}.$$

We endow $\pi(e)$ with the induced two-dimensional Lebesgue measure, that is we define the integration over $\pi(e)$ by the formula

$$\int_{\pi(e)} g(y) dy := \int_0^{+\infty} d\rho \int_{\mathbb{R}} dh g(\rho e + h\nu). \quad (9.4)$$

DEFINITION 9.6: NONLOCAL DIRECTIONAL CURVATURE | *We define the nonlocal directional curvature of ∂E at the point $p \in \partial E$ in direction e the quantity*

$$K_{s,e} := \int_{\pi(e)} \frac{|y' - p'|^{N-2} \tilde{\chi}_E(y)}{|y-p|^{N+2s}} dy. \quad (9.5)$$

¹Notice that $\pi(e)$ is simply the portion of the two-dimensional plane spanned by e and ν given by the vectors with positive scalar product with respect to e . We point out that a change of the orientation of ν does not change $\pi(e)$ which is therefore uniquely defined. Needless to say, such two-dimensional plane plays an important role even in the classical setting.

Without loss of generality, we can consider now a normal frame of coordinates in which p coincides with the origin 0 of \mathbb{R}^N , and the tangent space of S at 0 is the horizontal hyperplane $\{x_N = 0\}$. In this way we can take also

$$\nu = (0, \dots, 0, 1). \quad (9.6)$$

With this choice, (9.3) and (9.5) becomes

$$H_s = \frac{1}{\omega_{N-2}} \int_{\mathbb{R}^N} \frac{\tilde{\chi}_E(x)}{|x|^{N+2s}} dx \quad (9.7)$$

and

$$K_{s,e} = \int_{\pi(e)} \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} dy. \quad (9.8)$$

As a matter of fact, since the function $\tilde{\chi}_E(x)/|x|^{N+2s}$ is not in the space $L^1(\mathbb{R}^N)$, the integral in (9.7) has to be taken in the principal value sense, that is

$$\lim_{\varepsilon \downarrow 0} \int_{CB_\varepsilon} \frac{\tilde{\chi}_E(x)}{|x|^{N+2s}} dx. \quad (9.9)$$

Similarly, the integral in (9.8) may be taken in the principal value sense as

$$\lim_{\varepsilon \downarrow 0} \int_{\pi(e) \setminus B_\varepsilon} \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} dy. \quad (9.10)$$

Next observation points out that these definitions are well-posed, in view of the smoothness of ∂E :

LEMMA 9.7 | *The limits² in (9.9) and (9.10) exist and are finite.*

We postpone the proof of Lemma 9.7 to Section 9.8. Though the definition of the nonlocal direction curvature may look rather mysterious at a first glance, it finds a concrete justification thanks to the following result:

THEOREM 9.8 | *In the setting above*

$$H_s = \int_{S^{N-2}} K_{s,e} d\mathcal{H}^{N-2}(e).$$

Namely, Theorem 9.8 states that the nonlocal mean curvature is the average of the nonlocal directional curvatures, thus providing a nonlocal counterpart of Theorem 9.4. See Section 9.3 for the proof of Theorem 9.8.

In the particular case when the set E is characterized as the subgraph of a function $f \in C^2(\mathbb{R}^{N-1})$ (that, due to our normalization setting, satisfies $f(0) = 0$ and $\nabla f(0) = 0$), namely if $E = \{x_N < f(x')\}$, then formula (9.8) can be written directly in terms of f , according to the expression

$$K_{s,e} = 2 \int_0^{+\infty} d\rho \rho^{N-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}}. \quad (9.11)$$

The proof of (9.11) is deferred to Section 9.4.

REMARK 9.9 | We point out that if f is a radial function, i.e. $f(\rho e) = \phi(\rho)$ for some $\phi : [0, +\infty) \rightarrow \mathbb{R}$, equation (9.11) becomes

$$\begin{aligned} K_{s,e} &= 2 \int_0^{+\infty} d\rho \frac{1}{\rho^{2+2s}} \int_0^{\phi(\rho)} \frac{dh}{(1 + \rho^{-2}h^2)^{s+N/2}} = 2 \int_0^{+\infty} d\rho \frac{1}{\rho^{1+2s}} \int_0^{\frac{\phi(\rho)}{\rho}} \frac{d\tau}{(1 + \tau^2)^{s+N/2}} = \\ &= \int_0^{+\infty} \frac{1}{\rho^{1+2s}} \cdot F\left(\frac{\phi(\rho)}{\rho}\right) d\rho, \end{aligned}$$

²Similar statements can be found in [17, Theorem 5.1], where less regularity is asked on E and only the limsup is calculated, and [53, Lemma 1]; we add here the finiteness of the nonlocal directional curvature.

where

$$F(t) := \int_0^t (1 + \tau^2)^{-s-N/2} d\tau.$$

We observe that the function F is nonlinear, thus the nonlocal directional curvature depends on the graph of the set in a nonlinear fashion. Comparing this remark with Theorem 9.1, we notice that this phenomenon is in sharp contrast with the classical case.

REMARK 9.10 | In our setting $K_{s,-e}$ is, in general, not equal to $K_{s,e}$, differently from the classical case in which $K_{-e} = K_e$. For a notion of fractional directional curvature that is even on S^{N-2} one can consider $\tilde{K}_{s,e} := (K_{s,e} + K_{s,-e})/2$. Of course the results presented in this paper hold for $\tilde{K}_{s,e}$ too (with obvious minor modifications).

9.2.2 The context in which nonlocal curvatures naturally come forth

We now give some further motivation for the study of curvatures of nonlocal type. A few years ago the notion of *s-minimal set* has been introduced, see [17]. Roughly speaking, one can think of the problem of minimizing functionals which have a strong nonlocal flavor, meaning that these functionals take into account power-like interactions between distant objects. In particular, one is interested in the functional

$$\mathcal{J}(A, B) = \frac{1}{\omega_{N-1}} \int_A \int_B \frac{dx dy}{|x - y|^{N+2s}}$$

for every measurable sets³ $A, B \subseteq \mathbb{R}^N$, and in the minimization of the functional

$$\text{Per}_s(E, U) := \mathcal{J}(E \cap U, U \setminus E) + \mathcal{J}(E \cap U, \mathcal{C}E \cap CU) + \mathcal{J}(E \setminus U, U \setminus E),$$

called the *s-perimeter of E in U*, where $E, U \subseteq \mathbb{R}^N$ are measurable sets and U is bounded.

A set $E_\star \subseteq \mathbb{R}^N$ that minimizes $\text{Per}_s(E, U)$ among all the measurable sets $E \subseteq \mathbb{R}^N$ such that $E \setminus U = E_\star \setminus U$ is called *s-minimal*. In this framework, U can be viewed as an ambient space, meaning the space in which one is free to modify the set E , while the shape of E is fixed outside U and $E \setminus U$ plays the role of a boundary datum.

As the reader may have noticed, the notation $\text{Per}_s(E, U)$ and the name “*s-perimeter*” strongly remind the notation $\text{Per}(E, U)$ for the perimeter of a set E in U , see [46] (and indeed *s-minimal* sets are the natural nonlocal generalizations of sets with minimal perimeter). For instance, it is proved in [21, 4] that, as $s \uparrow \frac{1}{2}$, the *s-perimeter* reduces to the classical perimeter, namely

$$\lim_{s \uparrow \frac{1}{2}} (1 - 2s) \text{Per}_s(E, B_r) = \text{Per}(E, B_r) \quad \text{for a.e. } r > 0. \quad (9.12)$$

Also, asymptotics of the *s-perimeter* as $s \downarrow 0$ are studied in [38].

While, in the classical setting, sets with minimal perimeter satisfy the zero mean curvature equation, it is proved in [17] that if E_\star is an *s-minimal* set and $p \in \partial E$, then

$$\int_{\mathbb{R}^N} \frac{\tilde{\chi}_{E_\star}(x)}{|x - p|^{N+2s}} dx = 0. \quad (9.13)$$

Of course this equation makes sense if ∂E_\star is smooth enough near p , so in general [17] has to deal with (9.13) in a suitable weak (and in fact viscosity) sense. In this setting, one can say that (9.13) is the Euler-Lagrange equation of the functional Per_s and so, by analogy with the classical case, it is natural to consider the left hand side of (9.13) as a nonlocal mean curvature.

This justifies Definition 9.5. Furthermore, in [53, 19] a nonlocal approximation scheme of motion by mean curvature has been developed. This scheme differs from the classical one recalled in Section 9.1 since it substitutes the standard heat equation in (9.2) with its nonlocal counterpart

$$\begin{cases} \partial_t u(x, t) = -(-\Delta)^s u(x, t) \\ u(x, 0) = \tilde{\chi}_\Omega(x). \end{cases}$$

³We refer here to sets as in [22, footnote 1].

With this modification it has been proved that the counterpart of the normal velocity at a point $x_0 \in \partial\Omega$ is given by the quantity (see [17] and references therein)

$$\int_{\mathbb{R}^N} \frac{\tilde{\chi}_\Omega(x)}{|x - x_0|^{N+2s}} dx. \quad (9.14)$$

9.2.3 Some comparisons between classical and nonlocal directional curvatures

Now we turn to the study of the objects that we have introduced in the last paragraph, by stating some properties. Our goal is threefold: first we study the directions in which maximal curvatures are attained, then we are interested in asymptotics for $s \uparrow 1/2$, finally we present an example dealing with the relation between the nonlocal mean curvature and the average of extremal nonlocal directional curvatures.

First of all, we establish that the counterparts of Theorem 9.1 and Remark 9.2 do not hold in the nonlocal framework. Indeed, the direction that maximizes the nonlocal directional curvature is not, in general, orthogonal to the one that minimizes it. Even more, one can prescribe arbitrarily the set of directions that maximize and minimize the nonlocal directional curvature, according to the following result:

THEOREM 9.11: DIRECTIONS OF EXTREMAL NONLOCAL CURVATURES | *For any two disjoint, nonempty, closed subsets*

$$\Sigma_-, \Sigma_+ \subseteq S^{N-2},$$

there exists a set $E \subseteq \mathbb{R}^N$ such that ∂E is C^2 , $0 \in \partial E$ and

$$K_{s,e_-} < K_{s,e} < K_{s,e_+}, \quad \text{for any } e_- \in \Sigma_-, e \in S^{N-2} \setminus (\Sigma_+ \cup \Sigma_-), e_+ \in \Sigma_+,$$

and the minimum and maximum of the nonlocal directional curvatures are attained at any point of Σ_- and Σ_+ respectively.

We remark that, in the statement above, it is not necessary to assume any smoothness on the boundary of the sets Σ_- and Σ_+ in S^{N-2} .

Next result points out that the definition of nonlocal directional curvature is consistent with the classical concept of directional curvature and reduces to it in the limit:

THEOREM 9.12: ASYMPTOTICS TO 1/2 | *For any $e \in S^{N-2}$*

$$\lim_{s \uparrow \frac{1}{2}} (1 - 2s) K_{s,e} = K_e \quad (9.15)$$

and

$$\lim_{s \uparrow \frac{1}{2}} (1 - 2s) H_s = H,$$

where K_e (resp., H) is the directional curvature of E in direction e (resp., the mean curvature of E) at 0 .

Notice that Theorem 9.12 can be seen as an extension of the asymptotics in (9.12) for the directional and mean curvatures.

A further remark is that, differently from the local case, in the nonlocal one it is not possible to calculate the mean curvature simply by taking the arithmetic mean of the principal curvatures (this in dimension $N = 3$ reduces to the half of the sum between the maximal and the minimal directional curvatures). This phenomenon is a consequence of Theorem 9.12 and it may also be detected by an explicit example:

EXAMPLE 9.13 | Let $E = \{(x, y, z) \in \mathbb{R}^3 : z \leq 8x^2y^2\}$. Let H_s be the nonlocal mean curvature at $0 \in \partial E$. Let also $K_{s,e}$ be the nonlocal curvature at 0 in direction e ,

$$\lambda_- := \min_{e \in S^1} K_{s,e} \quad \text{and} \quad \lambda_+ := \max_{e \in S^1} K_{s,e}.$$

Then λ_- is attained at $(0, 1)$, λ_+ is attained at $(\sqrt{2}/2, \sqrt{2}/2)$, and $H_s \neq (\lambda_- + \lambda_+)/2$.

Of course, Example 9.13 is in sharp contrast with the classical case, recall Remark 9.2. For the proof of the claims related to Example 9.13 see Section 9.7.

9.3 PROOF OF THEOREM 9.8

Given a function G , we apply (9.4) to the function $g(y) := |y'|^{N-2}G(y)$. For this, we recall the normal coordinates in (9.6) and, with a slight abuse of notation we identify the vector $e = (e_1, \dots, e_{N-1}, 0) \in \mathbb{R}^N$ with $(e_1, \dots, e_{N-1}) \in \mathbb{R}^{N-1}$, so that we write

$$\pi(e) \ni y = \rho e + h\nu = (\rho e, h). \quad (9.16)$$

Then (9.4) reads

$$\int_{\pi(e)} dy |y'|^{N-2} G(y) = \int_0^{+\infty} d\rho \int_{\mathbb{R}} dh \rho^{N-2} G(\rho e, h).$$

We integrate this identity over $e \in S^{N-2}$: by recognizing the polar coordinates in \mathbb{R}^{N-1} we obtain

$$\begin{aligned} \int_{S^{N-2}} d\mathcal{H}^{N-2}(e) \int_{\pi(e)} dy |y'|^{N-2} G(y) &= \int_{\mathbb{R}} dh \int_{S^{N-2}} d\mathcal{H}^{N-2}(e) \int_0^{+\infty} d\rho \rho^{N-2} G(\rho e, h) = \\ &= \int_{\mathbb{R}} dh \int_{\mathbb{R}^{N-1}} dx' G(x', h) = \int_{\mathbb{R}^N} dx G(x). \end{aligned}$$

We apply this formula to $G(y) := \tilde{\chi}_E(y)/|y|^{N+2s}$ and we recall (9.7) and (9.8), so to conclude that

$$\int_{S^{N-2}} d\mathcal{H}^{N-2}(e) K_{s,e} = \int_{S^{N-2}} d\mathcal{H}^{N-2}(e) \int_{\pi(e)} dy \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} = \int_{\mathbb{R}^N} dx \frac{\tilde{\chi}_E(x)}{|x|^{N+2s}} = \omega_{N-2} H_s,$$

establishing Theorem 9.8. \square

9.4 PROOF OF (9.11)

We exploit again the notation in (9.16) and (9.4) applied to $g(y) := |y'|^{N-2} \tilde{\chi}_E(y)/|y|^{N+2s}$, to see that

$$\int_{\pi(e)} dy \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} = \int_0^{+\infty} d\rho \int_{\mathbb{R}} dh \frac{\rho^{N-2} \tilde{\chi}_E(\rho e, h)}{(\rho^2 + h^2)^{s+N/2}}. \quad (9.17)$$

Now we observe that $\tilde{\chi}_E(\rho e, h) = 1$ if $h < -|f(\rho e)|$ and $\tilde{\chi}_E(\rho e, h) = -1$ if $h > |f(\rho e)|$, being E the subgraph of f . Therefore, for any fixed $e \in S^{N-2}$ the map

$$h \mapsto \frac{\rho^{N-2} \tilde{\chi}_E(\rho e, h)}{(\rho^2 + h^2)^{s+N/2}}$$

is odd for $h \in \mathbb{R} \setminus [-|f(\rho e)|, |f(\rho e)|]$ and therefore

$$\int_{\mathbb{R} \setminus [-|f(\rho e)|, |f(\rho e)|]} dh \frac{\rho^{N-2} \tilde{\chi}_E(\rho e, h)}{(\rho^2 + h^2)^{s+N/2}} = 0. \quad (9.18)$$

The subgraph property also gives that

$$\begin{aligned} \int_{-|f(\rho e)|}^{|f(\rho e)|} dh \frac{\rho^{N-2} \tilde{\chi}_E(\rho e, h)}{(\rho^2 + h^2)^{s+N/2}} &= \begin{cases} 2 \int_0^{|f(\rho e)|} dh \frac{\rho^{N-2}}{(\rho^2 + h^2)^{s+N/2}} & \text{if } f(\rho e) \geq 0, \\ -2 \int_{-|f(\rho e)|}^0 dh \frac{\rho^{N-2}}{(\rho^2 + h^2)^{s+N/2}} & \text{if } f(\rho e) \leq 0 \end{cases} \\ &= 2 \int_0^{|f(\rho e)|} dh \frac{\rho^{N-2}}{(\rho^2 + h^2)^{s+N/2}}. \end{aligned}$$

This, (9.17) and (9.18) give that

$$\int_{\pi(e)} dy \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} = 2 \int_0^{+\infty} d\rho \int_0^{f(\rho e)} dh \frac{\rho^{N-2}}{(\rho^2 + h^2)^{s+N/2}},$$

and so (9.11) follows now from (9.8). \square

9.5 PROOF OF THEOREM 9.11

In \mathbb{R}^N define the set $E = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N \leq f(x')\}$. We will construct f in such a way to make it a regular function, say at least C^2 . For this, we fix two closed and disjoint sets Σ_- and Σ_+ in S^{N-2} , and we take $a \in C^\infty(S^{N-2}, [0, 1])$ in such a way that

$$\begin{aligned} a(e) &= 0 && \text{for any } e \in \Sigma_-, \\ a(e) &\in (0, 1) && \text{for any } e \in S^{N-2} \setminus (\Sigma_- \cup \Sigma_+), \\ a(e) &= 1 && \text{for any } e \in \Sigma_+. \end{aligned} \quad (9.19)$$

The existence of such an a is warranted by a strong version of the smooth Urysohn Lemma (notice that $a \in C^\infty(S^{N-2})$ in spite of the fact that no regularity assumption has been taken on Σ_- and Σ_+ and that a takes values 0 and 1 only in $\Sigma_- \cup \Sigma_+$). We provide the details of the construction of a for the facility of the reader. For this we observe that Σ_- is a closed set in \mathbb{R}^{N-1} . So, by Theorem 1.1.4 in [58], there exists $f_- \in C^\infty(\mathbb{R}^{N-1})$ such that $f_-(p) = 0$ for any $p \in \Sigma_-$ and $f_-(p) \neq 0$ for any $p \in \mathbb{R}^{N-1} \setminus \Sigma_-$. Then the function $g_-(p) := (f_-(p))^2$ satisfies that $g_-(p) = 0$ for any $p \in \Sigma_-$ and $g_-(p) > 0$ for any $p \in \mathbb{R}^{N-1} \setminus \Sigma_-$. Similarly, there exists $g_+ \in C^\infty(\mathbb{R}^{N-1})$ such that $g_+(p) = 0$ for any $p \in \Sigma_+$ and $g_+(p) > 0$ for any $p \in \mathbb{R}^{N-1} \setminus \Sigma_+$. Then the function

$$\mathbb{R}^{N-1} \ni p \mapsto a(p) := \frac{g_-(p)}{g_+(p) + g_-(p)}$$

satisfies (9.19) as desired.

Now we take an even function $\phi \in C_0^\infty(\mathbb{R}, [0, 1])$ with $\phi(\rho) > 0$ if $\rho \in (1, 2)$,

$$\lim_{\rho \rightarrow 0^+} \phi''(\rho) = \lim_{\rho \rightarrow 0^+} \phi'(\rho) = \lim_{\rho \rightarrow 0^+} \phi(\rho) = 0.$$

Then we define f using the polar coordinates of \mathbb{R}^{N-1} , namely we set

$$f(x') := a(e)\phi(\rho), \quad \text{where } \rho = |x'| \text{ and } e = \frac{x'}{|x'|}.$$

By construction f is C^2 in the whole of \mathbb{R}^{N-1} (in particular, in a neighborhood of 0). Also

$$0 = a(e_-)\phi(\rho) \leq \phi(\rho)a(e) \leq \phi(\rho)a(e_+) \text{ for all } \rho > 0, e \in S^{N-2} \quad (9.20)$$

whenever we choose $e_- \in \Sigma_-$, $e_+ \in \Sigma_+$ and $e \in S^{N-2} \setminus (\Sigma_- \cup \Sigma_+)$, and strict inequalities occur whenever $\rho \in (1, 2)$. Therefore, by (9.11) and (9.20),

$$\begin{aligned} K_{s, e_-} = 0 &= 2 \int_0^{+\infty} d\rho \rho^{N-2} \int_0^{\phi(\rho)a(e_-)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} \leq \\ &\leq 2 \int_0^{+\infty} d\rho \rho^{N-2} \int_0^{\phi(\rho)a(e)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} = K_{s, e} \end{aligned}$$

and

$$\begin{aligned} K_{s, e} &= 2 \int_0^{+\infty} d\rho \rho^{N-2} \int_0^{\phi(\rho)a(e)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} \leq \\ &\leq 2 \int_0^{+\infty} d\rho \rho^{N-2} \int_0^{\phi(\rho)a(e_+)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} = K_{s, e_+}, \end{aligned}$$

hence $K_{s,e}$ attains its minimum at any point of Σ_- and its maximum at any point of Σ_+ (and only there). \square

9.6 PROOF OF THEOREM 9.12

For simplicity, we consider here the case in which E is a subgraph, namely that there exists $f \in C^2(\mathbb{R}^{N-1})$, such that $f(0) = 0$, $\nabla f(0) = 0$ and $E = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N \leq f(x')\} \subseteq \mathbb{R}^N$. Such assumption can be easily dropped a posteriori just working in local coordinates and observing that the contribution to $K_{s,e}$ coming from far is bounded uniformly⁴ when $s \uparrow 1/2$ and so it does not contribute to the limit in (9.15).

So, to prove Theorem 9.12, we take equation (9.11) and split the integral in $\eta > 0$

$$K_{s,e} = 2 \int_0^\eta d\rho \rho^{N-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} + 2 \int_\eta^{+\infty} d\rho \rho^{N-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}}. \quad (9.21)$$

Let us start from the second addendum:

$$\begin{aligned} & \left| \int_\eta^{+\infty} d\rho \rho^{N-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} \right| \leq \int_\eta^{+\infty} d\rho \int_0^{+\infty} dh \frac{\rho^{N-2}}{(\rho^2 + h^2)^{s+N/2}} \leq \\ & \leq \int_\eta^{+\infty} d\rho \int_0^{+\infty} dh \frac{(\rho^2 + h^2)^{(N-2)/2}}{(\rho^2 + h^2)^{s+N/2}} \leq \int_{\mathbb{R}^2 \setminus B_\eta} |x|^{-2-2s} dx = 2\pi \int_\eta^{+\infty} r^{-1-2s} dr = \frac{\pi}{s} \eta^{-2s}. \end{aligned} \quad (9.22)$$

Now we look at the first addendum in (9.21): for this we write, for $\eta \in (0, 1)$ sufficiently small,

$$D_e^2 f(0) \frac{\rho^2}{2} - \varepsilon(\rho) \leq f(\rho e) \leq D_e^2 f(0) \frac{\rho^2}{2} + \varepsilon(\rho), \quad \rho \in (0, \eta) \quad (9.23)$$

where $\varepsilon : (0, \eta) \rightarrow (0, +\infty)$ is defined as

$$\varepsilon(\rho) := \sup_{|\xi'| \leq \rho} |D_e^2 f(\xi') - D_e^2 f(0)| \frac{\rho^2}{2}.$$

Let also

$$E(\eta) := \sup_{0 < \rho \leq \eta} \frac{\varepsilon(\rho)}{\rho^2} = \frac{1}{2} \sup_{|\xi'| \leq \eta} |D_e^2 f(\xi') - D_e^2 f(0)|.$$

and observe that

$$E(\eta) \downarrow 0 \quad \text{as } \eta \downarrow 0. \quad (9.24)$$

Then, if we denote by $D := \frac{1}{2} D_e^2 f(0)$, we have that

$$\begin{aligned} & \left| \int_0^\eta d\rho \rho^{N-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} - \frac{D}{1-2s} \right| \leq \left| \int_0^1 d\rho \rho^{N-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} - \frac{D}{1-2s} \right| \\ & + \left| \int_0^\eta d\rho \rho^{N-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} - \int_0^\eta d\rho \rho^{N-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} \right| \\ & + \left| \int_\eta^1 d\rho \rho^{N-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} \right|. \end{aligned} \quad (9.25)$$

⁴In further detail, recalling (9.4),

$$\begin{aligned} & \left| \int_{\pi(e) \setminus B_1} \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} dy \right| \leq \int_{\pi(e)} \frac{|y|^{N-2} \chi_{CB_1}(y)}{|y|^{N+2s}} dy \leq \\ & \leq \int_0^{+\infty} d\rho \int_{-\infty}^{+\infty} dh \chi_{\mathbb{R} \setminus (-1,1)}(\rho^2 + h^2) (\rho^2 + h^2)^{-1-s} \leq \int_{\mathbb{R}^2 \setminus B_1} dx |x|^{-2-2s} = 2\pi \int_1^{+\infty} dr r^{-1-2s} = \frac{\pi}{s} \end{aligned}$$

that is uniformly bounded as $s \uparrow 1/2$. This means that we can suppose that ∂E is a graph in, say, B_1 and replace it outside B_1 without affecting the statement of Theorem 9.12.

The latter term is uniformly bounded as $s \uparrow 1/2$: indeed

$$\begin{aligned} \left| \int_{\eta}^1 d\rho \rho^{N-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} \right| &\leq \int_{\eta}^1 d\rho \rho^{N-2} \cdot \frac{|D|\rho^2}{\rho^{N+2s}} = \int_{\eta}^1 d\rho \frac{|D|}{\rho^{2s}} \\ &= |D| \frac{1 - \eta^{1-2s}}{1 - 2s} \xrightarrow{s \uparrow \frac{1}{2}} -|D| \ln \eta, \end{aligned} \quad (9.26)$$

and therefore, for s close to $1/2$,

$$\left| \int_{\eta}^1 d\rho \rho^{N-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} \right| \leq -\tilde{D} \ln \eta, \quad (9.27)$$

where \tilde{D} is a positive constant depending only on D .

The central term in (9.25) may be estimated using (9.23): indeed

$$\begin{aligned} &\left| \int_0^{\eta} d\rho \rho^{N-2} \int_0^{f(\rho e)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} - \int_0^{\eta} d\rho \rho^{N-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} \right| \\ &\leq \int_0^{\eta} d\rho \rho^{N-2} \left| \int_{D\rho^2}^{f(\rho e)} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} \right| \leq \int_0^{\eta} \rho^{N-2} \frac{|f(\rho e) - D\rho^2|}{\rho^{N+2s}} d\rho \\ &\leq \int_0^{\eta} \frac{\varepsilon(\rho)}{\rho^{2+2s}} d\rho \leq E(\eta) \int_0^{\eta} \rho^{-2s} d\rho = E(\eta) \cdot \frac{\eta^{1-2s}}{1-2s}. \end{aligned} \quad (9.28)$$

It remains now to estimate the first term on the right hand side of (9.25). For this we apply the change of variable $t = h/\rho$ and we see that

$$\begin{aligned} \left| \rho^{N-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} - D\rho^{-2s} \right| &= |D|\rho^{-2s} \left| \frac{1}{D\rho^2} \int_0^{D\rho^2} \frac{dh}{\left(1 + \frac{h^2}{\rho^2}\right)^{s+N/2}} - 1 \right| \\ &= |D|\rho^{-2s} \left| \frac{1}{D\rho} \int_0^{D\rho} \frac{dt}{(1+t^2)^{s+N/2}} - 1 \right|. \end{aligned}$$

Now, since the map $t \mapsto 1/(1+t^2)^{s+N/2}$ is decreasing, we have that

$$\frac{1}{(1+D^2\rho^2)^{s+N/2}} \leq \int_0^{|D|\rho} \frac{dt}{(1+t^2)^{s+N/2}} \leq 1.$$

Using this and a Taylor expansion, we obtain

$$\begin{aligned} \left| \rho^{N-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} - D\rho^{-2s} \right| &= |D|\rho^{-2s} \left(1 - \frac{1}{|D|\rho} \int_0^{|D|\rho} \frac{dt}{(1+t^2)^{s+N/2}} \right) \\ &\leq |D|\rho^{-2s} \left(1 - \frac{1}{(1+D^2\rho^2)^{s+N/2}} \right) \leq |D|\rho^{-2s} \cdot (\beta D^2\rho^2) = \beta |D|^3 \rho^{2-2s} \end{aligned}$$

for any $\rho \in (0, 1)$, where β is a suitable positive constant only depending on N . By integrating this estimate over the interval $(0, 1)$ we conclude that

$$\begin{aligned} \left| \int_0^1 d\rho \rho^{N-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} - \frac{D}{1-2s} \right| &= \\ &= \left| \int_0^1 d\rho \left[\rho^{N-2} \int_0^{D\rho^2} \frac{dh}{(\rho^2 + h^2)^{s+N/2}} - D\rho^{-2s} \right] \right| \leq \tilde{\beta}, \end{aligned} \quad (9.29)$$

for a suitable $\tilde{\beta}$ possibly depending on D and N , but independent of s .

Resuming all this calculations in one formula and putting together the information in equations (9.21), (9.22), (9.25), (9.27), (9.28) and (9.29), we obtain that

$$\left| K_{s,e} - \frac{2D}{1-2s} \right| \leq \frac{2\pi}{s} \eta^{-2s} - \tilde{D} \ln \eta + 2E(\eta) \cdot \frac{\eta^{1-2s}}{1-2s} + 2\tilde{\beta}, \quad (9.30)$$

hence

$$\limsup_{s \uparrow \frac{1}{2}} (1-2s) \left| K_{s,e} - \frac{2D}{1-2s} \right| \leq 2E(\eta).$$

But, by (9.24), $E(\eta)$ is arbitrarily small. Hence we can conclude

$$\lim_{s \uparrow \frac{1}{2}} (1-2s) \left| K_{s,e} - \frac{2D}{1-2s} \right| = 0$$

which also implies

$$\lim_{s \uparrow \frac{1}{2}} (1-2s) K_{s,e} = 2D = D_e^2 f(0),$$

that is the desired claim. Moreover, since estimate (9.30) is uniform on S^{N-2} , we have also the convergence

$$(1-2s)H_s = \frac{1-2s}{\omega_{N-2}} \int_{S^{N-2}} K_{s,e} d\mathcal{H}^{N-2}(e) \xrightarrow{s \uparrow \frac{1}{2}} \frac{1}{\omega_{N-2}} \int_{S^{N-2}} D_e^2 f(0) d\mathcal{H}^{N-2}(e) = H.$$

This completes the proof of Theorem 9.12. \square

9.7 PROOF OF THE CLAIMS IN EXAMPLE 9.13

In \mathbb{R}^3 define

$$E = \{(x, y, z) \in \mathbb{R}^3 : z \leq f(x, y) = 8x^2y^2\}$$

and denote by (ρ, θ, z) the cylindrical coordinates, i.e.

$$\rho = \sqrt{x^2 + y^2}, \quad \cos \theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin \theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

In this way, we set $e := (\cos \theta, \sin \theta) \in S^1$ and we have that $(x, y) = \rho e$. Notice that

$$1 - \cos 4\theta = 1 - \cos^2 2\theta + \sin^2 2\theta = 2 \sin^2 2\theta = 8 \sin^2 \theta \cos^2 \theta \cdot \frac{\rho^4}{\rho^4} = \frac{8x^2y^2}{(x^2 + y^2)^2}$$

therefore the function f , written in terms of (ρ, θ) , becomes

$$f(x, y) = f(\rho e) = \tilde{f}(\rho, \theta) = (1 - \cos 4\theta) \rho^4.$$

With a slight abuse of notation, we denote by $K_{s,\theta}$ the nonlocal directional curvature of E at 0 in direction $e = (\cos \theta, \sin \theta)$ (i.e. $K_{s,\theta}$ is short for $K_{s,(\cos \theta, \sin \theta)}$). We recall that, in this case, we can compute nonlocal curvatures exploiting identity (9.11), i.e.

$$K_{s,\theta} = 2 \int_0^{+\infty} d\rho \int_0^{\tilde{f}(\rho, \theta)} dz \frac{\rho}{(\rho^2 + z^2)^{s+3/2}}. \quad (9.31)$$

In the above domain of integration it holds that

$$0 \leq z \leq \tilde{f}(\rho, \theta) = (1 - \cos 4\theta) \rho^4$$

and so

$$\rho \geq \sqrt[4]{\frac{z}{1 - \cos 4\theta}}, \quad \text{when } \cos 4\theta \neq 1.$$

Therefore, integrating first in the ρ variable, we deduce from (9.31) that

$$\begin{aligned} K_{s,\theta} &= \int_0^{+\infty} dz \int_{\sqrt[4]{\frac{z}{1 - \cos 4\theta}}}^{+\infty} d\rho \frac{2\rho}{(\rho^2 + z^2)^{s+3/2}} = \int_0^{+\infty} dz \frac{(\rho^2 + z^2)^{-s-1/2}}{-s - \frac{1}{2}} \Big|_{\rho=\sqrt[4]{\frac{z}{1 - \cos 4\theta}}}^{\rho=+\infty} \\ &= \frac{2}{2s+1} \int_0^{+\infty} \frac{dz}{\left(z^2 + \sqrt{\frac{z}{1 - \cos 4\theta}}\right)^{s+1/2}} = \frac{2}{2s+1} \int_0^{+\infty} \frac{(1 - \cos 4\theta)^{s/2+1/4}}{(z^2 \sqrt{1 - \cos 4\theta} + \sqrt{z})^{s+1/2}} dz. \end{aligned} \quad (9.32)$$

We concentrate now on maximal and minimal nonlocal directional curvatures. Since $K_{s,\theta}$ is a nonnegative quantity (because $\tilde{f}(\rho, \theta)$ is a nonnegative function) and since $K_{s,0} = K_{s,\pi/2} = 0$, then $K_{s,\theta}$ attains its minimum in 0 and $\pi/2$. Also, $\tilde{f}(\rho, \pi/4) \geq \tilde{f}(\rho, \theta)$ for every positive ρ and $\theta \in [0, \pi]$, thus $K_{s,\theta}$ attains its maximum for $\theta = \pi/4$. On the one hand we have that the arithmetic mean of the maximal and minimal nonlocal principal curvatures is given by

$$\frac{K_{s,0} + K_{s,\pi/4}}{2} = \frac{1}{2} K_{s,\pi/4} = \frac{2^{s/2+1/4}}{2s+1} \int_0^{+\infty} \frac{dz}{(\sqrt{2}z^2 + \sqrt{z})^{s+1/2}}, \quad (9.33)$$

thanks to (9.32).

On the other hand, the nonlocal mean curvature is

$$H_s = \frac{1}{2\pi} \int_0^{2\pi} K_{s,\theta} d\theta$$

and we are going to estimate this quantity, in order to show that it is not equal to the arithmetic mean of the principal nonlocal curvatures (i.e. to the quantity in (9.33)). Note that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} K_{s,\theta} d\theta &= \frac{1}{(2s+1)\pi} \int_0^{2\pi} d\theta \int_0^{+\infty} dz \frac{(1 - \cos 4\theta)^{s/2+1/4}}{(z^2 \sqrt{1 - \cos 4\theta} + \sqrt{z})^{s+1/2}} \\ &\geq \frac{1}{(2s+1)\pi} \int_0^{2\pi} (1 - \cos 4\theta)^{s/2+1/4} d\theta \cdot \int_0^{+\infty} \frac{dz}{(\sqrt{2}z^2 + \sqrt{z})^{s+1/2}} \\ &= \frac{1}{\pi 2^{s/2+1/4}} \int_0^{2\pi} (1 - \cos 4\theta)^{s/2+1/4} d\theta \cdot \underbrace{\frac{2^{s/2+1/4}}{2s+1} \int_0^{+\infty} \frac{dz}{(\sqrt{2}z^2 + \sqrt{z})^{s+1/2}}}_{=K_{s,\pi/4}/2} \\ &= \frac{1}{\pi 2^{s/2+1/4}} \int_0^{2\pi} (1 - \cos 4\theta)^{s/2+1/4} d\theta \cdot \frac{1}{2} K_{s,\pi/4}. \end{aligned} \quad (9.34)$$

On the other hand, with the substitution $\varphi := 4\theta$ and recalling that $s + 1/2 \in (1/2, 1)$, we see that

$$\begin{aligned} \frac{1}{\pi 2^{s/2+1/4}} \int_0^{2\pi} (1 - \cos 4\theta)^{s/2+1/4} d\theta &= \frac{1}{4\pi} \int_0^{8\pi} \left(\sqrt{\frac{1 - \cos \varphi}{2}} \right)^{s+1/2} d\varphi = \frac{1}{\pi} \int_0^{2\pi} \left(\sin \frac{\varphi}{2} \right)^{s+1/2} d\varphi \\ &\geq \frac{1}{\pi} \int_0^{2\pi} \sin \frac{\varphi}{2} d\varphi = -\frac{2}{\pi} \cos \frac{\varphi}{2} \Big|_0^{2\pi} = \frac{4}{\pi} > 1. \end{aligned}$$

By combining this with (9.33) and (9.34) we conclude that

$$H_s \geq \frac{1}{2} K_{s,\pi/4} = \frac{K_{s,0} + K_{s,\pi/4}}{2}.$$

This establishes the claims in Example 9.13.

9.8 PROOF OF LEMMA 9.7

We prove now (9.10), while (9.9) can be deduced with minor modifications with respect to the proof below. Let

$$\sigma_\varepsilon := \int_{\pi(e) \setminus B_\varepsilon} \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} dy.$$

Notice that, for any $\varepsilon > \varepsilon' > 0$,

$$|\sigma_{\varepsilon'} - \sigma_\varepsilon| = \left| \int_{\pi(e) \cap (B_\varepsilon \setminus B_{\varepsilon'})} \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} dy \right|. \quad (9.35)$$

Since ∂E is C^2 , in normal coordinates we may suppose that, for small $\varepsilon > 0$,

$$E \cap B_\varepsilon \subseteq \{x = (x', x_N) \in \mathbb{R}^N : x_N \leq M|x'|^2\} \quad \text{and} \quad (CE) \cap B_\varepsilon \subseteq \{x \in \mathbb{R}^N : x_N \geq -M|x'|^2\},$$

for a suitable $M > 0$. This provides a cancellation of the contributions outside the set $E_{\varepsilon, \varepsilon'} := \{x \in B_\varepsilon \setminus B_{\varepsilon'} : x_N \leq M|x'|^2\}$, namely

$$\begin{aligned} \left| \int_{\pi(e) \cap (B_\varepsilon \setminus B_{\varepsilon'})} \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} dy \right| &= \left| \int_{\pi(e) \cap E_{\varepsilon, \varepsilon'}} \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} dy \right| \leq \\ &\leq \int_{\pi(e) \cap E_{\varepsilon, \varepsilon'}} \frac{|y'|^{N-2}}{|y|^{N+2s}} dy \leq \int_{\pi(e)} \frac{\chi_{E_{\varepsilon, \varepsilon'}}(y)}{|y'|^{2+2s}} dy, \end{aligned}$$

since $|y| \geq |y'|$. Note now that in $E_{\varepsilon, \varepsilon'}$ we have

$$|x'|^2 \geq |\varepsilon'|^2 - |x_N|^2 \geq |\varepsilon'|^2 - M^2|x'|^4 \geq |\varepsilon'|^2 - M^2\varepsilon^2|x'^2|$$

and therefore

$$|x'|^2 \geq \frac{|\varepsilon'|^2}{1 + M^2\varepsilon^2} =: |\varepsilon^*|^2$$

That is, by (9.4),

$$\left| \int_{\pi(e) \cap (B_\varepsilon \setminus B_{\varepsilon'})} \frac{|y'|^{N-2} \tilde{\chi}_E(y)}{|y|^{N+2s}} dy \right| \leq \int_{\varepsilon^*}^\varepsilon d\rho \int_{-M\rho^2}^{M\rho^2} dh \rho^{-2-2s} \leq 2M \int_{\varepsilon^*}^\varepsilon d\rho \rho^{-2s} = \frac{2M(\varepsilon - \varepsilon^*)^{1-2s}}{1-2s}$$

which is infinitesimal with ε (recall that $s \in (0, 1/2)$ by assumption). This and (9.35) imply that for any $\varepsilon_N \downarrow 0$, σ_{ε_N} is a Cauchy sequence, as desired. \square

Bibliography

- [1] Nicola Abatangelo. Large s -harmonic functions and boundary blow-up solutions for the fractional laplacian. *Discr. Cont. Dyn. Syst. A*, 35(12):5555–5607, 2015.
- [2] Nicola Abatangelo. On a fractional Keller-Osserman condition. *preprint at arXiv:1412.6298*, (Dec. 2014).
- [3] Nicola Abatangelo and Enrico Valdinoci. A notion of nonlocal curvature. *Numer. Funct. Anal. Optim.*, 35(7-9):793–815, 2014.
- [4] Luigi Ambrosio, Guido De Philippis, and Luca Martinazzi. γ -convergence of nonlocal perimeter functionals. *Manuscripta Math.*, 134(3-4):377–403, 2011.
- [5] Sheldon Axler, Paul Bourdon, and Wade Ramey. *Harmonic function theory*, volume 137 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001.
- [6] Catherine Bandle. Asymptotic behavior of large solutions of elliptic equations. *An. Univ. Craiova Ser. Mat. Inform.*, 32:1–8, 2005.
- [7] Guy Barles, Emmanuel Chasseigne, and Cyril Imbert. On the Dirichlet problem for second-order elliptic integro-differential equations. *Indiana Univ. Math. J.*, 57(1):213–246, 2008.
- [8] Begoña Barrios, Alessio Figalli, and Enrico Valdinoci. Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 5, 2012.
- [9] Peter W. Bates. On some nonlocal evolution equations arising in materials science. In *Nonlinear dynamics and evolution equations*, volume 48 of *Fields Inst. Commun.*, pages 13–52. Amer. Math. Soc., Providence, RI, 2006.
- [10] Ludwig Bieberbach. $\Delta u = e^u$ und die automorphen Funktionen. *Mathematische Annalen*, 77(2):173–212, 1916.
- [11] Krzysztof Bogdan. The boundary Harnack principle for the fractional Laplacian. *Studia Math.*, 123(1):43–80, 1997.
- [12] Krzysztof Bogdan. Representation of α -harmonic functions in Lipschitz domains. *Hiroshima Math. J.*, 29(2):227–243, 1999.
- [13] Krzysztof Bogdan, Tomasz Byczkowski, Tadeusz Kulczycki, Michal Ryznar, Renming Song, and Zoran Vondraček. *Potential analysis of stable processes and its extensions*, volume 1980 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Edited by Piotr Graczyk and Andrzej Stos.
- [14] Matteo Bonforte, Yannick Sire, and Juan L. Vázquez. Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains. *Discr. Cont. Dyn. Syst. A*, 35(12):5725–5767, 2015.

- [15] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.
- [16] Xavier Cabré and Jinggang Tan. Positive solutions of nonlinear problems involving the square root of the Laplacian. *Adv. Math.*, 224(5):2052–2093, 2010.
- [17] Luis Caffarelli, Jean-Michel Roquejoffre, and Ovidiu Savin. Nonlocal minimal surfaces. *Comm. Pure Appl. Math.*, 63(9):1111–1144, 2010.
- [18] Luis Caffarelli and Luis Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.
- [19] Luis Caffarelli and Panagiotis Souganidis. Convergence of nonlocal threshold dynamics approximations to front propagation. *Arch. Ration. Mech. Anal.*, 195(1):1–23, 2010.
- [20] Luis Caffarelli and Pablo Stinga. Fractional elliptic equations, Caccioppoli estimates and regularity. *preprint at arXiv:1409.7721*, (Jan. 2015).
- [21] Luis Caffarelli and Enrico Valdinoci. Uniform estimates and limiting arguments for nonlocal minimal surfaces. *Calc. Var. Partial Differential Equations*, 41(1-2):203–240, 2011.
- [22] Luis Caffarelli and Enrico Valdinoci. Regularity properties of nonlocal minimal surfaces via limiting arguments. *Adv. Math.*, 248:843–871, 2013.
- [23] Antonio Capella, Juan Dávila, Louis Dupaigne, and Yannick Sire. Regularity of radial extremal solutions for some non-local semilinear equations. *Comm. Partial Differential Equations*, 36(8):1353–1384, 2011.
- [24] M. Cristina Caputo and Nestor Guillen. Regularity for non-local almost minimal boundaries and applications. *preprint at arXiv:1003.2470*, (June 2011).
- [25] Huyuan Chen, Patricio Felmer, and Alexander Quaas. Large solutions to elliptic equations involving the fractional Laplacian. *preprint*, 2013.
- [26] Huyuan Chen and Laurent Véron. Semilinear fractional elliptic equations involving measures. *J. Differential Equations*, 257(5):1457–1486, 2014.
- [27] Zhen-Qing Chen. Multidimensional symmetric stable processes. *Korean J. Comput. Appl. Math.*, 6:227–266, 1999.
- [28] Philippe Clément and Guido Sweers. Getting a solution between sub- and supersolutions without monotone iteration. *Rend. Istit. Mat. Univ. Trieste*, 19(2):189–194, 1987.
- [29] Rama Cont and Peter Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.
- [30] Ovidiu Costin and Louis Dupaigne. Boundary blow-up solutions in the unit ball: asymptotics, uniqueness and symmetry. *J. Differential Equations*, 249(4):931–964, 2010.
- [31] Ovidiu Costin, Louis Dupaigne, and Olivier Goubet. Uniqueness of large solutions. *J. Math. Anal. Appl.*, 395(2):806–812, 2012.
- [32] Edward B. Davies. The equivalence of certain heat kernel and Green function bounds. *J. Funct. Anal.*, 71(1):88–103, 1987.
- [33] Edward B. Davies and Barry Simon. Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.*, 59(2):335–395, 1984.

-
- [34] Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez. A general fractional porous medium equation. *Comm. Pure Appl. Math.*, 65(9):1242–1284, 2012.
 - [35] Jean-Stéphane Dherzin and Jean-François Le Gall. Wiener’s test for super-Brownian motion and the Brownian snake. *Probab. Theory Related Fields*, 108(1):103–129, 1997.
 - [36] Abdelwaheb Dhifli, Habib Mâagli, and Malek Zribi. On the subordinate killed BM in bounded domains and existence results for nonlinear fractional Dirichlet problems. *Math. Ann.*, 352(2):259–291, 2012.
 - [37] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.
 - [38] Serena Dipierro, Alessio Figalli, Giampiero Palatucci, and Enrico Valdinoci. Asymptotics of the s -perimeter as $s \searrow 0$. *Discr. Contin. Dyn. Syst.*, 33(7):2777–2790, 2013.
 - [39] Manfredo P. do Carmo. *Differential geometry of curves and surfaces*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1976. Translated from the Portuguese.
 - [40] Serge Dumont, Louis Dupaigne, Olivier Goubet, and Vicentiu Rădulescu. Back to the Keller-Osserman condition for boundary blow-up solutions. *Adv. Nonlinear Stud.*, 7(2):271–298, 2007.
 - [41] Louis Dupaigne. *Stable solutions of elliptic partial differential equations*, volume 143 of *Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2011.
 - [42] Lawrence C. Evans. Convergence of an algorithm for mean curvature motion. *Indiana Univ. Math. J.*, 42(2):533–557, 1993.
 - [43] Patricio Felmer and Alexander Quaas. Boundary blow up solutions for fractional elliptic equations. *Asymptot. Anal.*, 78(3):123–144, 2012.
 - [44] Matthieu Felsinger, Moritz Kassmann, and Paul Voigt. The Dirichlet problem for nonlocal operators. *Math. Z.*, 279(3-4):779–809, 2015.
 - [45] Nicola Fusco, Vincent Millot, and Massimiliano Morini. A quantitative isoperimetric inequality for fractional perimeters. *J. Funct. Anal.*, 261(3):697–715, 2011.
 - [46] Enrico Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.
 - [47] Joseph Glover, Zoran R. Pop-Stojanovic, Murali Rao, Hrvoje Šikić, Renming Song, and Zoran Vondraček. Harmonic functions of subordinate killed Brownian motion. *J. Funct. Anal.*, 215(2):399–426, 2004.
 - [48] Joseph Glover, Murali Rao, Hrvoje Šikić, and Ren Ming Song. Γ -potentials. In *Classical and modern potential theory and applications*, volume 430 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 217–232. Kluwer Acad. Publ., Dordrecht, Chateau de Bonas, 1994.
 - [49] Maria del Mar González, Rafe Mazzeo, and Yannick Sire. Singular solutions of fractional order conformal laplacians. *Journal of Geometric Analysis*, 22(3):845–863, 2012.
 - [50] Gerd Grubb. Fractional Laplacians on domains, a development of Hörmander’s theory of μ -transmission pseudodifferential operators. *Adv. Math.*, 268:478–528, 2015.
 - [51] Gerd Grubb. Regularity of spectral fractional Dirichlet and Neumann problems. *preprint at arXiv:1412.3744*, (Feb. 2015).

- [52] Gerd Grubb. Boundary problems for mu-transmission pseudodifferential operators, including fractional Laplacians. *preprint at arXiv:1403.7140*, (Mar. 2014).
- [53] Cyril Imbert. Level set approach for fractional mean curvature flows. *Interfaces Free Bound.*, 11(1):153–176, 2009.
- [54] Hitoshi Ishii. A generalization of the Bence, Merriman and Osher algorithm for motion by mean curvature. In *Curvature flows and related topics (Levico, 1994)*, volume 5 of *GAKUTO Internat. Ser. Math. Sci. Appl.*, pages 111–127. Gakkōtoshō, Tokyo, 1995.
- [55] Kenneth H. Karlsen, Francesco Petitta, and Suleyman Ulusoy. A duality approach to the fractional Laplacian with measure data. *Publ. Mat.*, 55(1):151–161, 2011.
- [56] Joseph B. Keller. On solutions of $\Delta u = f(u)$. *Comm. Pure Appl. Math.*, 10:503–510, 1957.
- [57] Tomasz Klimsiak and Andrzej Rozkosz. Dirichlet forms and semilinear elliptic equations with measure data. *J. Funct. Anal.*, 265(6):890–925, 2013.
- [58] Steven G. Krantz and Harold R. Parks. *The geometry of domains in space*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, 1999.
- [59] Denis A. Labutin. Wiener regularity for large solutions of nonlinear equations. *Arkiv för Matematik*, 41(2):307–339, 2003.
- [60] N. S. Landkof. *Foundations of modern potential theory*. Springer-Verlag, New York, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.
- [61] Jean-François Le Gall. The brownian snake and solutions of $\delta u = u^2$ in a domain. *Probability Theory Related Fields*, 102(3):393–432, 1995.
- [62] J.-L. Lions and E. Magenes. *Problèmes aux limites non homogènes et applications. Vol. 1*. Travaux et Recherches Mathématiques, No. 17. Dunod, Paris, 1968.
- [63] Charles Loewner and Louis Nirenberg. Partial differential equations invariant under conformal or projective transformations. In *Contributions to Analysis (a collection of papers dedicated to Lipman Bers)*, pages 245–272. Academic Press, New York, 1974.
- [64] Moshe Marcus and Laurent Véron. Existence and uniqueness results for large solutions of general nonlinear elliptic equations. *J. of Evol. Eq.*, 3(1):637–652, 2004.
- [65] Moshe Marcus and Laurent Véron. *Nonlinear second order elliptic equations involving measures*, volume 21 of *De Gruyter Series in Nonlinear Analysis and Applications*. De Gruyter, Berlin, 2014.
- [66] Barry Merriman, James K. Bence, and Stanley J. Osher. Motion of multiple functions: a level set approach. *J. Comput. Phys.*, 112(2):334–363, 1994.
- [67] Marcelo Montenegro and Augusto C. Ponce. The sub-supersolution method for weak solutions. *Proc. Amer. Math. Soc.*, 136(7):2429–2438, 2008.
- [68] Benoît Mselati. Classification and probabilistic representation of the positive solutions of a semilinear elliptic equation. *Mem. Amer. Math. Soc.*, 168(798), 2004.
- [69] Edward Nelson. A proof of Liouville’s theorem. *Proc. Amer. Math. Soc.*, 12:995, 1961.
- [70] Robert Osserman. On the inequality $\Delta u \geq f(u)$. *Pacific J. Math.*, 7:1641–1647, 1957.
- [71] Augusto C. Ponce. Selected problems on elliptic equations involving measures. *arXiv:1204.0668*.

-
- [72] Marcel Riesz. Intégrales de Riemann–Liouville et potentiels. *Acta Sci. Math.*, 9:1–42, 1938.
 - [73] Xavier Ros-Oton and Joaquim Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9)*, 101(3):275–302, 2014.
 - [74] Xavier Ros-Oton and Joaquim Serra. The Pohozaev identity for the fractional Laplacian. *Arch. Ration. Mech. Anal.*, 213(2):587–628, 2014.
 - [75] Ovidiu Savin and Enrico Valdinoci. Regularity of nonlocal minimal cones in dimension 2. *Calc. Var. Partial Differential Equations*, 48(1-2):33–39, 2013.
 - [76] René L. Schilling, Renming Song, and Zoran Vondraček. *Bernstein functions*, volume 37 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2012. Theory and applications.
 - [77] Raffaella Servadei and Enrico Valdinoci. On the spectrum of two different fractional operators. *Proc. Roy. Soc. Edinburgh Sect. A*, 144(4):831–855, 2014.
 - [78] Luis Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.
 - [79] Yannick Sire and Enrico Valdinoci. Fractional Laplacian phase transitions and boundary reactions: a geometric inequality and a symmetry result. *J. Funct. Anal.*, 256(6):1842–1864, 2009.
 - [80] Dejan Slepčev. Approximation schemes for propagation of fronts with nonlocal velocities and Neumann boundary conditions. *Nonlinear Anal.*, 52(1):79–115, 2003.
 - [81] Renming Song. Sharp bounds on the density, Green function and jumping function of subordinate killed BM. *Probab. Theory Relat. Fields*, 128(2):606–628, 2004.
 - [82] Renming Song and Zoran Vondraček. Potential theory of subordinate killed Brownian motion in a domain. *Probab. Theory Related Fields*, 125(4):578–592, 2003.
 - [83] Guido Stampacchia. *Équations elliptiques du second ordre à coefficients discontinus*. Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965). Les Presses de l’Université de Montréal, Montreal, Que., 1966.
 - [84] J. F. Toland. The Peierls-Nabarro and Benjamin-Ono equations. *J. Funct. Anal.*, 145(1):136–150, 1997.
 - [85] Enrico Valdinoci. A fractional framework for perimeters and phase transitions. *Milan J. Math.*, 81(1):1–23, 2013.
 - [86] Juan Luis Vázquez. Recent progress in the theory of nonlinear diffusion with fractional Laplacian operators. *Discrete Contin. Dyn. Syst. Ser. S*, 7(4):857–885, 2014.
 - [87] Laurent Véron. Elliptic equations involving measures. In *Stationary partial differential equations, Vol. I*, Handb. Differ. Equ., pages 593–712. North-Holland, Amsterdam, 2004.
 - [88] Qi S. Zhang. The boundary behavior of heat kernels of Dirichlet Laplacians. *J. Differential Equations*, 182(2):416–430, 2002.